Field and Galois Theory 3

1) Let k be a field of characteristic $p > 0$, $\alpha_i \in \bar{k}(i = 1, 2, \cdots, n)$ with $\alpha_i^p \in k$. The set $\{\alpha_1, \cdots, \alpha_n\}$ is called **p-independent** over k if $[k(\alpha_1, \cdots, \alpha_n) : k] = p^n$. Let K/k be a field extension with $K^p \subseteq k$. A subset $\{\alpha_1, \cdots, \alpha_n\} \subseteq K$ of p-independent elements over k is called a **p-basis** if $K = k(\alpha_1, \cdots, \alpha_n)$.

   Let K/k be a finite purely inseparable extension. Show that the extension $K/k(K^p)$ has a p-basis.

2) For every $n \in \mathbb{N}$, give an example of a finite field extension of degree $\geq n$ which is neither separable nor purely inseparable.

3) Let k be a field of positive characteristic and K a finite extension of k. Prove that if the $[K : k]$ is prime to the characteristic of k, then K/k is separable.

4) Show that the inseparable closure of a field K in $\bar{K}$ is the smallest intermediate subfield between K and $\bar{K}$ that is perfect.

5) Let k be a field of characteristic $p > 0$ and K/k a purely inseparable extension. The ***p*-exponent** of the extension K/k is defined to be

$$\inf\{n \in \mathbb{N} | \alpha^{p^n} \in k, \forall \alpha \in K\} \in \mathbb{N} \cup \{+\infty\}.$$

   Show that:

   (a) If K/k is finite, then the p-exponent of K over the separable closure of k in K is smaller than or equal to $\log_p[K : k]_i$.

   (b) Give an example of a finite purely inseparable extension such that its p-exponent is strictly smaller than the cardinality of a p-basis[1].

[1] We know by problem 1 that all p-bases have the same cardinality.

(c) Give an example of an infinite purely inseparable extension with finite p-exponent.

(d) Give an example of a purely inseparable extension with infinite p-exponent.

6) Let K/k be algebraic and L/k an algebraic extension such that K and L are contained in some field. Show that:

(a) if L/k is purely inseparable, then

$$[KL : L]_s = [K : k]_s$$

(b) if K/k is finite and L/k is finite and separable, then

$$[KL : L]_i = [K : k]_i.$$

7) Let K be a Galois extension of F and let $a \in K$. Let $n = [K : F]$, $r = [F(a) : F]$, and $H = \mathrm{Gal}(K/F(a))$. Let $\tau_1, \ldots, \tau_r$ be left coset representatives of H in G. Show that

$$\min(F, a) = \prod_{i=1}^{r} (x - \tau_i(a)) .$$

Conclude that

$$\prod_{\sigma \in \mathrm{Gal}(K/F)} (x - \sigma(a)) = \min(F, a)^{n/r}.$$

8) Let K be a finite Galois extension of F with Galois group G. Let L be an intermediate extension, and let H be the corresponding subgroup of G. If $N(H)$ is the normalizer of H in G, let L_0 be the fixed field of $N(H)$. Show that L/L_0 is Galois and that if M is any subfield of L containing F for which L/M is Galois, then M contains L_0.

9) Prove the *Normal Basis Theorem*: If K is a finite Galois extension of F, then there is an $a \in K$ such that $\{\sigma(a) : \sigma \in \mathrm{Gal}(K/F)\}$ is a basis for K as an F-vector space.