Field and Galois Theory 2

1) Let F/E be an algebraic extension, $\sigma : E \to L$ an embedding with L algebraically closed and algebraic over E^σ, prove that

$$|\operatorname{Hom}_\sigma(F, \bar{L})| = [F : E]_s$$

2) Let E be a finite field and F/E an algebraic extension. Prove that F/E is separable.

3) Let F be an algebraic extension of E. Show that every subring of F which contains E is actually a field. Is this necessary true if F is not algebraic over E? Prove or give a counter example.

4) Let F be a field and $\alpha \in \bar{F}$. Show that if α is not separable over F, then $\exists\, n \geq 1$, such that α^{p^n} is separable over F, where $p = \operatorname{char} F$.

5) Let F be a field of characteristic $p > 0$. Prove that if $\alpha \in F \setminus F^p$ (where $F^p = \{x^p \,|\, x \in F\}$), then for all $n \geq 1$, the polynomial $X^{p^n} - \alpha$ is irreducible in $F[X]$.

6) Give 3 (different) examples of an algebraic extension which is neither normal nor separable.

7) A field k is called **perfect** if any algebraic extension K/k is separable. Show that a field k of characteristic $p > 0$ is perfect if and only if the **Frobenius** morphism

$$\operatorname{Frob}_p : k \to k, \quad x \mapsto x^p$$

is a ring isomorphism.