Field and Galois Theory 1

In this homework set, E is a field, F is an extension field of E, and $E[X]$ is the polynomial ring in variable X and coefficients in E.

Let $\mathcal{C}$ be a certain class of extension fields $F \subset E$. We shall say that $\mathcal{C}$ is **distinguished** if it satisfies the following conditions:

(i) Let $k \subset F \subset E$ be a tower of fields. The extension $k \subset E$ is in $\mathcal{C}$ if and only if $k \subset F$ is in $\mathcal{C}$ and $F \subset E$ is in $\mathcal{C}$.

(ii) If $k \subset E$ is in $\mathcal{C}$, if F is any extension of k, and E, F are both contained in some field, then $F \subset EF$ is in $\mathcal{C}$.

(iii) If $k \subset F$ and $k \subset E$ are in $\mathcal{C}$ and F, E are subfields of a common field, then $k \subset FE$ is in $\mathcal{C}$.

1) Prove that the class of finite extensions is distinguished.

2) Prove that algebraically closed fields are infinite (as a set).

3) Find a counterexample to show that the class of normal extensions is NOT distinguished.

4) Assume $f(X) := X^n - a \in E[X]$ is irreducible, m divides n and α is a root of f in an algebraic closure of E. Prove $[E(\alpha^m) : E] = n/m$. What is $\min(\alpha^m; E)(X)$?

5) Let K, L be two finite extensions of a field k, contained in some field. Show that

$$[KL : k] \leq [K : k][L : k].$$

If $[K : k]$ and $[L : k]$ are relatively prime, show that one has an equality sign in the above relation.

6) Let $f(x) \in E[X]$ be a polynomial of degree n. Let K be its splitting field. Show that $[K : E]$ divides $n!$.

7) Find the splitting field of $X^{p^8} - 1$ over the field $\mathbb{Z}/p\mathbb{Z}$.

8) Let α be a real number such that $\alpha^4 = 5$. Show that:

   (a) $\mathbb{Q}(i\alpha^2)$ is normal over $\mathbb{Q}$;

   (b) $\mathbb{Q}(\alpha + i\alpha)$ is NOT normal over $\mathbb{Q}$.

9) Describe the splitting fields of the following polynomials over $\mathbb{Q}$, and find the degree of each such splitting field.

   (a) $X^2 - 1$

   (b) $X^3 - 2$

   (c) $(X^3 - 2)(X^2 - 2)$

   (d) $X^6 + X^3 + 1$

   (e) $X^5 - 7$