A Brief Introduction to Homotopy Theory

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Abstract

In this article, we study the elementary and basic notions of homotopy theory such as cofibrations, fibrations, weak equivalences etc. and some of their properties. For technical reasons and to simplify the arguments, we suppose that all spaces are compactly generated. The results may hold in more general conditions, however, we do not intend to introduce them with minimal hypothesis. We would omit the proof of certain theorems and proposition, either because they are straightforward or because they demand more information and technics of algebraic topology which do not lie within the scope of this article. We would also suppose some familiarity of elementary topology. A more detailed and further considerations can be found in the bibliography at the end of this article.

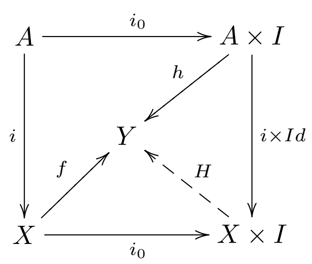
Contents

1 Cofibrations

In this and the following two chapters, we elaborate the fundamental tools and definitions of our study of homotopy theory, cofibrations, fibrations and exact homotopy sequences.

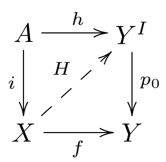
Definition 1.1. *A map* $i : A \longrightarrow X$ *is a* cofibration *if it satisfies the homotopy extension property (HEP), i.e. , given a map* $f : X \longrightarrow Y$ *and a homotopy* $h : A \times I \longrightarrow Y$ *whose restriction to* $A \times \{0\}$ *is* $f \circ i$*, there exists an extension of* h *to* $X \times I$*.*

This situation is expressed schematically as follows:



where $i_0(u) = (u, 0)$.

We may write this property in an other equivalent way. $i : A \longrightarrow X$ is a cofibration, if there exists a lifting H in the following diagram



where $p_0(\beta) = \beta(0)$.

Remark 1.2. We do not require H to be unique, and it is usually not the case.

Remark 1.3. Some authors suppose that $i : A \longrightarrow X$ is an inclusion with closed image. We will show that this can be derived from the definition, however we need some definitions first.

Definition 1.4. *The* mapping cylinder *of* $f : X \longrightarrow Y$ *is defined to be the pushout of the maps* $f : X \longrightarrow Y$ *and* $i_0 : X \longrightarrow X \times I$*, and we note it* M_f*, so* $M_f \equiv Y \cup_f (X \times I)$*.*

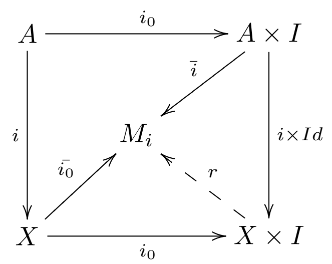
We often construct new spaces and new maps from the given spaces and maps, and one way of such a construction is to take the pushout of two maps. The following proposition states that the class of cofibrations is closed under taking pushouts. Thus we may take pushout of two maps without any restriction.

Proposition 1.5. *Suppose that $i : A \longrightarrow X$ is a cofibration and $g : A \longrightarrow B$ is any map, then the induced map $B \longrightarrow B \cup_g X$ is a cofibration.*

There is another proposition that is useful to making new cofibration from ones given.

Proposition 1.6. *If $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ are cofibrations then so is $i \times j : A \times B \longrightarrow X \times Y$.*

In the definition of a cofibration we asked too much, and in fact it suffices to have the HEP for the following diagram.



If there exists such a map, and if we are given a map $f : X \longrightarrow Y$ and a homotopy $h : A \times I \longrightarrow Y$, the universal property of pushout implies that there is a map $M_i \longrightarrow Y$ whose composition with r is a map that extends h and its restriction to X is f. Thus we have the following proposition.

Proposition 1.7. *A map $i : A \longrightarrow X$ is a cofibration if and only if there exists a filler in the last diagram.*

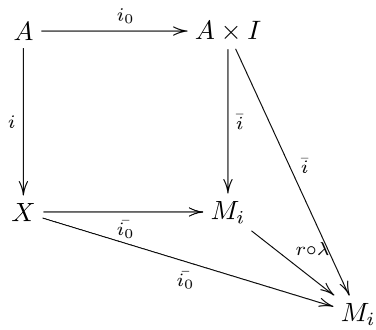
Example 1.8. The inclusion $i_s : Y \longrightarrow Y \times I$ given by $i_s(y) = (y, s)$ is a cofibration.

Now we are able to prove the following proposition.

Proposition 1.9. *If $i : A \longrightarrow X$ is a cofibration then it is an inclusion with closed image.*

PROOF. Consider the previous diagram. Observe that $\bar{i}$ is injective on $A \times (0, 1]$ since M_i can be regarded as $X \times \{0\} \bigcup A \times I/\{i(a) = (a, 0)\}$. Now if $i(a) = i(b)$

then $r(i(a),1) = r(i(b),1)$, so $\bar{i}(a,1) = \bar{i}(b,1)$, thus $a = b$. This shows that i is injective. For the second part, we first show that M_i is a closed subspace of $X \times I$. Define $\lambda : M_i \longrightarrow X \times I$, to be the identity on $X \times I$ and $\lambda(a,t) = (i(a),t)$ for all $a \in A, t \in I$. Clearly it is a well defined map. Now the composition $r \circ \lambda$ is a map that fills the pushout diagram



The uniqueness of such a map implies that it is the identity map, so λ is injective and we can identify M_i with its image. Since r is surjective, we have $M_i = r(X \times I) = \{z \in X \times I \mid r(z) = z\} =: \Delta$. In order to show that Δ is a closed subspace of $X \times I$, we have to show that its intersection with each compact subset of $X \times I$ is closed. So suppose that C is compact in $X \times I$, and $y \in C \setminus \Delta \cap C$, so $r(y)$ is different from y. Since C is Hausdorff, there exist open subsets V_1 and V_2 such that,

$$y \in C \cap V_1, r(y) \in C \cap V_2, C \cap V_1 \cap V_2 = \emptyset,$$

which implies that $y \in r^{-1}(V_2)$. Now put $U := V_1 \cap r^{-1}(V_2) \cap C$. It is an open subset of C that contains y and we have $U \cap \Delta = \emptyset$, for if $z \in U \cap \Delta$ we have $r(z) = z \in U$ so $r(z) \in V_1 \cap V_2 \cap C = \emptyset$ which is a contradiction. So $C \setminus \Delta \cap C$ is open in C, and so $\Delta \cap C$ is a closed subset of C. Now consider the inclusion $X \times \{1\} \xrightarrow{\iota} M_i$, $\iota^{-1}(M_i)$ is closed in $X \times \{1\}$ and we have $\iota^{-1}(M_i) = A \times \{1\}$, thus A is closed in X. □

An important property of cofibrations is that every map is a cofibration up to homotopy equivalence, more precisely if $f : X \longrightarrow Y$ is a map we can write it as the composite $X \xrightarrow{j} M_f \xrightarrow{r} Y$ where $j(x) = (x,1)$ and $r(y) = y$ on Y and $r(x,s) = f(x)$ on $X \times I$.

Proposition 1.10. *In the above notation j is a cofibration and r is a homotopy equivalence.*

There is a characterization theorem for cofibrations, but we need the following definition before.

Definition 1.11. *A pair (X, A) with $A \subseteq X$ is an* NDR-pair *(NDR for Neighbourhood Deformation Retract) if there is a map $u : X \longrightarrow I$ such that $u^{-1}(I) = A$ and a homotopy $h : X \times I \longrightarrow X$ such that $h_0 = Id, h(a, t) = a$ for all $a \in A$ and $t \in I$ and $h_1(x) \in A$ if $u(x) < 1$. (X, A) is a* DR-pair *if $u(x) < 1$ for all $x \in X$ in which case A is a deformation retract of X.*

Proposition 1.12. *If (X, A) and (Y, B) are NDR-pairs, then $(X \times Y, X \times B \cup A \times Y)$ is an NDR-pair. If (X, A) or (Y, B) is a DR-pair, then so is $(X \times Y, X \times B \cup A \times Y)$.*

Now we have the characterization theorem

Theorem 1.13. *Let A be a closed subset of X. The following statements are equivalent:*

(i) *(X, A) is an NDR-pair*

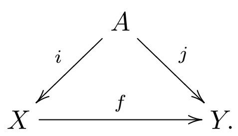
(ii) *$(X \times I, X \times \{0\} \cup A \times I)$ is a DR-pair*

(iii) *$X \times \{0\} \cup A \times I$ is a retract of $X \times I$*

(iv) *The inclusion $i : A \longrightarrow X$ is a cofibration*

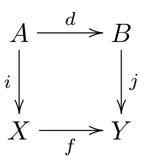
Sometimes it is important to work in the category of spaces under a given space,so define this category as follows.

Definition 1.14. *A* space under A *is a map $i : A \longrightarrow X$. A map of spaces under A is a commutative diagram*



A homotopy between maps f and $f^{'}$ under A is a homotopy h that at each time t is map under A, i.e., $h(i(a), t) = j(a)$ for all $a \in A$ and $t \in I$. We write $h : f \simeq f^{'}$ rel A. It gives rise to a notion of a homotopy equivalence under A, called a cofiber homotopy equivalence.

Proposition 1.15. *Assume given a commutative diagram*



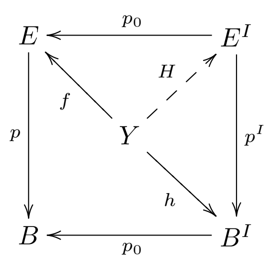
in which i and j are cofibrations and d and f are homotopy equivalences. Then $(f,d) : (X,A) \longrightarrow (Y,B)$ is a homotopy equivalence of pairs, i.e., there are homotopy inverses e of d and g of f such that $g \circ j = i \circ e$ together with homotopies $H : g \circ f \simeq Id$ and $K : f \circ g \simeq Id$, that extend homotopies $h : e \circ d \simeq Id$ and $k : d \circ e \simeq Id$. In particular if f is homotopy equivalence between two spaces under A, then it is a cofiber homotopy equivalence.

2 Fibrations

In this chapter we dualize the notions and theory of the last chapter.

Definition 2.1. *A surjective map $p : E \longrightarrow B$ is a* fibration *if it satisfies the covering homotopy property, i.e., given a map $f : Y \longrightarrow E$ and a homotopy $h : Y \times I \longrightarrow B$, there exists a lifting of h to E,whose restriction to $Y \times \{0\}$ is f.*

Remark 2.2. As we have seen for cofibrations there is an equivalent definition of a fibration in which we can better see the dualization.



where $p_0(\beta) = \beta(0)$

Remark 2.3. Again we do not require the uniqueness of such a lifting.

The class of fibrations is closed under the base extensions, i.e.,

Proposition 2.4. *Pullbacks of fibrations are fibrations.*

Definition 2.5. *Let $p : E \longrightarrow B$ be a map. Its* mapping path space *is a pullback of p and $p_0 : B^I \longrightarrow B$ and we note it $N_p \equiv E \times_p B^I = \{(e, \beta) \mid \beta(0) = p(e)\}$.*

The mapping path space is the dual of mapping cylinder and we will see that it plays the same role for the fibrations.

Definition 2.6. *Let $p : E \longrightarrow B$ be a map and let N_p be its mapping path space. A map $s : N_p \longrightarrow E^I$ such that $s(e, \beta)(0) = e$ and $p \circ s(e, \beta) = \beta$ is called* path lifting function.

We have seen that for a map $i : A \longrightarrow X$ to be a cofibration it suffices to admit a homotopy extension for its mapping cylinder. It turns out that for a map $p : E \longrightarrow B$ to be fibration it suffices to have a homotopy lifting for its mapping path space, or equivalently we have

Proposition 2.7. *A map $p : E \longrightarrow B$ is a fibration if and only if it admits a path lifting function.*

Proof. Replace Y by N_p in the test diagram of the equivalent definition of fibrations; necessity is then clear. So suppose that we have a path lifting function $s : N_p \longrightarrow E^I$ and maps $f : Y \longrightarrow E$ and $h : Y \longrightarrow B^I$. There is an induced map $g : Y \longrightarrow N_p$, since N_p is a pullback. The composite $s \circ g$ gives the required homotopy lifting. □

As an application of this proposition, we have the following example

Example 2.8. If $p : E \longrightarrow B$ is a covering, then p is a fibration with a unique path lifting function.

Example 2.9. The evaluation map $p_s : B^I \longrightarrow B$ given by $p_s(\beta) = \beta(s)$ is a fibration.

The relation between fibrations and cofibrations is stated in the following proposition

Proposition 2.10. *If $i : A \longrightarrow X$ is a cofibration and B is a space then the induced map $p = B^i : B^X \longrightarrow B^A$ is a fibration.*

Proof. It is an easy task to show that we have the following homeomorphisms

$$B^{M_i} = B^{X\times\{0\}\cup A\times I} \cong B^X \times_p (B^A)^I = N_p$$

If $r : X \times I \longrightarrow M_i$ is a retraction given by Theorem 1.13 then

$$B^r : N_p \cong B^{M_i} \longrightarrow B^{X\times I} \cong (B^X)^I$$

is a path lifting function, for if $(e, \beta) \in N_p$ the homeomorphism sends (e, β) to $\gamma \in B^{M_i}$, the function that is e on X and $\tilde{\beta}$ on $A \times I$, where $\tilde{\beta}(a, t) = \beta(t)(a)$, so $B^r(e, \beta)(x, 0) = (e, \beta)(r(x, 0)) = (e, \beta)(x) = e(x)$, so we have $B^r(e, \beta)(0) = e$ through the identification $B^{X\times I} \cong (B^X)^I$. We have also

$$p \circ B^r(e, \beta)(t)(a) = B^r(e, \beta)(i(a), t) = (e, \beta)(r(i(a), t)) = (e, \beta)(a, t) = \beta(t)(a)$$

hence B^r is a fibration. □

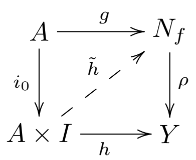
We have seen that every map can be factored as cofibration followed by a homotopy equivalence. We can dualize this property and get the following proposition,which will be of great use later.

Proposition 2.11. *Any map can be factored as a homotopy equivalence followed by a fibration.*

PROOF. Given a map$f : X \longrightarrow Y$, define $\upsilon : X \longrightarrow N_f$, by $\upsilon(x) = (x, c_{f(x)})$ where $c_{f(x)}$ is the constant path at $f(x)$ and define, $\rho : N_f \longrightarrow Y$, by $\rho(x, \beta) = \beta(1)$. We have $f = \rho \circ \upsilon$. Now let $\pi : N_f \longrightarrow X$ be the projection, then $\pi \circ \upsilon =$id and id $\simeq \upsilon \circ \pi$ via the homotopy $h : N_f \times I \longrightarrow N_f$ by setting

$$h(x, \beta)(t) = (x, \beta_t), \text{where} \beta_t(s) = \beta((1 - t)s)$$

Now suppose given a test diagram



Write $g(a) = (g_1(a), g_2(a))$ and set $\tilde{h}(a, t) = (g_1(a), j(a, t))$ where

$$j(a, t) = \begin{cases} g_2(a)(s + st) & \text{if} \quad 0 < s < 1/(1 + t) \\ h(a, s + st - 1) & \text{if} \quad 1/(1 + t) \leq s \leq 1 \end{cases}$$

So ρ satisfies the covering homotopy property,and thus it is a fibration. □

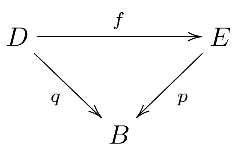
In Example 2.8 we have seen that a covering is a fibration, so we may think of fibrations as a generalization of coverings. This idea leads to a local criterion that allows us to recognize fibrations when we see them. But we need a definition before stating it.

Definition 2.12. *A* numerable open cover *of a space B is an open cover $\mathscr{O}$ such that for each $U \in \mathscr{O}$ there are continuous maps $\lambda_U : B \longrightarrow I$ such that $\lambda_U^{-1}(0, 1] = U$ and that the cover is locally finite, i.e., each $b \in B$ has an open neighbourhood that intersects finitely many $U \in \mathscr{O}$.*

Theorem 2.13. *Let $p : E \longrightarrow B$ be a map and let $\mathscr{O}$ be a numerable open cover of B. Then p is a fibration if and only if $p : p^{-1}(U) \longrightarrow U$ is a fibration for every $U \in \mathscr{O}$.*

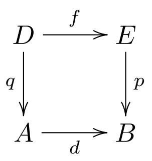
Now we give the definition dual to that of cofiber homotopy equivalence.

Definition 2.14. *A* space over *B is a map $p : E \longrightarrow B$. A* map *of spaces over B is a commutative diagram*

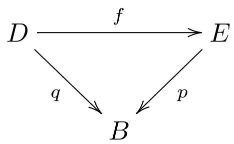


A homotopy *between maps of spaces over B is a homotopy that at each time t is a map over B. A homotopy equivalence over B is called* a fiber homotopy equivalence.

Proposition 2.15. *Assume given a commutative diagram*

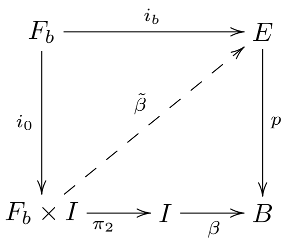


in which p and q are fibrations and d and f are homotopy equivalences. Then $(f, d) : q \longrightarrow p$ is a homotopy equivalence of fibrations. In particular if $f : D \longrightarrow E$ is a homotopy equivalence and we have a commutative diagram



then f is a fiber homotopy equivalence.

The statement means that there are homotopy inverses e of d and g of f such that $q \circ g = e \circ p$ together with homotopies $H : g \circ f \simeq \mathrm{id}$ and $K : f \circ g \simeq \mathrm{id}$ that cover homotopies $h : e \circ d \simeq \mathrm{id}$ and $k : d \circ e \simeq \mathrm{id}$.
Using the covering homotopy property of fibrations, $p : E \longrightarrow B$ induces a translation of fibers along path classes and we will see that this change of fiber is in fact a homotopy equivalence. Let F_b be the fiber over $b \in B$, i.e. , the set $p^{-1}(\{b\})$, let $i_b : F_b \hookrightarrow E$ be the inclusion and let $\beta \in B^I$ be a path joining b to $b^{'}$. There exists a lift $\tilde{\beta}$ in the following diagram



since p is a fibration. We have $p \circ \tilde{\beta}(e, t) = \beta(t)$ so at time t we have a map, $\tilde{\beta}_t : F_b \longrightarrow F_{\beta(t)}$, and in particular we have $\tilde{\beta}_1 : F_b \longrightarrow F_{b'}$, and we note

$$\tau[\beta] \equiv [\tilde{\beta}_1] : F_b \longrightarrow F_{b'}$$

which is called *the translation along the path class* $[\beta]$. The following proposition confirms that $\tau[\beta]$ is in fact well defined.

Proposition 2.16. *Using the above notations, if* $\beta \simeq \beta^{'}$ *then* $\tilde{\beta} \simeq \tilde{\beta}'$ *and in particular* $\tilde{\beta}_1 \simeq \tilde{\beta}_1^{'}$. *Therefore the homotopy class of* $\tilde{\beta}_1$ *which is noted by* $\tau[\beta]$ *is independent of the choice of* β *in its path class.*

One can easily see that

$$\tau[c_b] = [\mathrm{id}] \quad \text{and} \quad \tau[\gamma \star \beta] = \tau[\gamma] \circ \tau[\beta] \quad \text{with} \quad \beta(1) = \gamma(0)$$

so $\tau[\beta]$ has the inverse $\tau[\beta^{-1}]$ and we have

Proposition 2.17. *If there exists* $\beta \in B^I$, *a path starting at* b *and ending at* $b^{'}$, *then* F_b *and* $F_{b'}$ *are homotopic equivalent. Therefore if* B *is path connected, any two fibers of* B *are homotopy equivalent.*

These translations are functorial in the sense of the following proposition

Proposition 2.18. *Let* p *and* q *be fibrations in the following commutative diagram*

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

and let $\alpha : I \longrightarrow A$ *be a path from* a *to* $a^{'}$. *Then the following diagram is commutative up to homotopy equivalence.*

$$\begin{array}{ccc} F_a & \xrightarrow{g} & F_{f(a)} \\ {\scriptstyle \tau[\alpha]}\downarrow & & \downarrow{\scriptstyle \tau[f\circ\alpha]} \\ F_{a'} & \xrightarrow[g]{} & F_{f(a')}. \end{array}$$

If further $h : f \simeq f'$ *and* $H : g \simeq g'$ *in the commutative diagram*

$$\begin{array}{ccc} D \times I & \xrightarrow{H} & E \\ {\scriptstyle q\times id}\downarrow & & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[h]{} & B \end{array}$$

then the following diagram commutes up to homotopy equivalence

$$\begin{array}{ccc} & F_a & \\ {\scriptstyle g} \swarrow & & \searrow {\scriptstyle g'} \\ F_{f(a)} & \xrightarrow[\tau[h(\alpha)]]{} & F_{f'(a)} \end{array}$$

where $h(a)(t) = h(a, t)$.

3 Homotopy Exact Sequences

In this section we associate two exact sequences of spaces to a given map. These sequences will play an important role in homotopy theory, and we will see that they are very useful for calculating higher homotopy groups. All spaces in this section are based spaces, and we shall write $*$ generically for the basepoints.

Definition 3.1. *For spaces X and Y, $[X,Y]$ denotes the set of based homotopy classes of based maps $X \longrightarrow Y$. This set has a natural basepoint, namely the homotopy class of the constant map from X to the basepoint of Y. $F(X,Y)$ denotes the subspace of Y^X consisting of the based maps, with the constant base map as basepoint.*

Definition 3.2. *The* wedge product *of X and Y is the pushout of two inclusions $* \longrightarrow X$ and $* \longrightarrow Y$ and is noted by $X \vee Y$. Explicitly, $X \vee Y = \{(x,y) \in X \times Y \mid x = * \quad or \quad y = *\}$. The* smash product *of X and Y is defined by*

$$X \wedge Y = X \times Y / X \vee Y.$$

Proposition 3.3. *Let X, Y, Z be three spaces:*

(i) *We have a natural homeomorphism*

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)).$$

(ii) *We can identify $[X,Y]$ with $\pi_0(F(X,Y))$,where $\pi_0(X)$ denotes the set of path components of X.*

Definition 3.4. *Let X be a space. The* cone *or* reduced cone *on X is $CX = X \wedge I$, where the basepoint of I is 1. More explicitly we have*

$$CX = X \times I/(\{*\} \times I \cup X \times \{1\}).$$

The unreduced cone *on X which is again noted by CX is defined by*

$$CX = X \times I / X \times \{1\}.$$

We can view S^1 as $I/\partial I$ and denote its basepoint by 1. We can now define some important objects of our study.

Definition 3.5. *The* suspension *or* reduced suspension *of a space X is*

$$\Sigma X = X \wedge S^1 = X \times S^1/(\{*\} \times S^1 \cup X \times \{1\}) = X \times I/(\{*\} \times I \cup X \times \partial I).$$

The unreduced suspension *of X is defined to be the quotient of $X \times I$ obtained by identifying the set $X \times \{1\}$ to a point and the set $X \times \{0\}$ to another point.*

We have the dual constructions and definitions as follows

Definition 3.6. *The* path space *of a space X is the based space $PX = F(I, X)$, where I is given the basepoint 0. Thus its elements are paths in X starting from the basepoint.*

Definition 3.7. *The* loop space *of a space X is the based space $\Omega X = F(S^1, X)$. Its points are loops at the basepoint of X.*

To see how these constructions are related one to another we have the following proposition

Proposition 3.8. *There is a natural homeomorphism*

$$F(\Sigma X, Y) \cong F(X, \Omega Y)$$

and a natural bijection

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

PROOF. To see the first statement define the function

$$^\sharp : F(\Sigma X, Y) \longrightarrow F(X, \Omega Y)$$

that sends each map $f : \Sigma X \longrightarrow Y$ to the map $f^\sharp : X \longrightarrow \Omega Y$ defined by $f^\sharp(x)(s) = f(x \wedge s)$. The other statement follows from Proposition 3.3 by taking π_0 of both side of the previous homeomorphism. $\square$

We can endow $[\Sigma X, Y]$ with a multiplication as follows. For $f, g : \Sigma X \longrightarrow Y$ define

$$([g] + [f])(x \wedge t) = (g^\sharp(x) \star f^\sharp(x))(t) = \begin{cases} f(x \wedge 2t) & \text{if} \quad 0 \leq t \leq 1/2 \\ g(x \wedge (2t-1)) & \text{if} \quad 1/2 \leq t \leq 1 \end{cases}$$

It is easy to see that this multiplication is well defined.

Proposition 3.9. $[\Sigma X, Y]$ *is a group and* $[\Sigma^2 X, Y]$ *is an Abelian group.*

Definition 3.10. *If in the definition of a cofibration we require that all maps and spaces be based, then we have a* based cofibration, *which we call cofibration throughout this section. We call a space X* nondegenerately based *or* well pointed *if the inclusion $* \longrightarrow X$ is a cofibration in the unbased sense.*

Remark 3.11. A based map $i : A \longrightarrow X$ that is a cofibration in the unbased sense is a cofibration in the based sense as well since, the basepoint of X lies in A.

Definition 3.12. *We write* Y_+ *for the union of a space* Y *and a disjoint basepoint. The space* $X \wedge I_+$ *is called the* reduced cylinder *on* X*. We write in this section* M_f *for the* based mapping cylinder $Y \cup_f (X \wedge I_+)$ *of a based map* $f : X \longrightarrow Y$*.*

Remark 3.13. Observe that we can identify $X \wedge I_+$ with $X \times I/\{*\} \times I$.

Remark 3.14. Observe also that a based homotopy $X \times I \longrightarrow Y$ is the same thing as a based map $X \wedge I_+ \longrightarrow Y$. As we have seen for the unbased case in Proposition 1.7, a map $i : A \longrightarrow X$ is a cofibration if and only if M_i is a retract of $X \wedge I_+$.

Definition 3.15. *For a based map* $f : X \longrightarrow Y$ *define the* homotopy cofiber *to be*

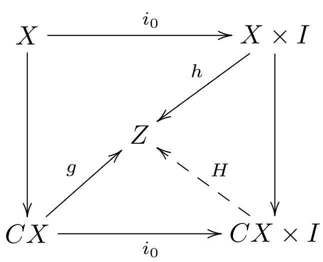
$$Cf = Y \cup_f CX = M_f/j(X)$$

where $j : X \longrightarrow M_f$ *sends* x *to* $(x, 1)$*.*

Remark 3.16. Quotienting out Y, we get ΣX from C_f, more precisely we have $C_f/Y \cong \Sigma X$.

Proposition 3.17. *The inclusions* $X \longrightarrow CX$ *that sends* x *to* $(x, 0)$ *and* $Y \longrightarrow C_f$ *are cofibrations.*

Proof. For the first inclusion consider a test diagram



define a homotopy $H : CX \times I \longrightarrow Z$ by

$$H((x,t),s) = \begin{cases} g(x, t + s(1+t)) & \text{if} \quad 0 \le s \le (1-t)/(1+t) \\ h(x, \frac{1}{2}(s(1+t) + t - 1)) & \text{if} \quad (1-t)/(1+t) \le s \le 1 \end{cases}$$

For the second inclusion, observe that it is the pushout of f and the previous cofibration. □

Definition 3.18. *A sequence* $S' \xrightarrow{f} S \xrightarrow{g} S''$ *of pointed sets is exact if* $g(s) = *$ *if and only if* $f(s') = s$ *for some* $s' \in S'$*.*

Definition 3.19. *Let $f : X \longrightarrow Y$ be a based map and $i : Y \longrightarrow C_f$ be the cofibration cited above. Let*

$$\pi : C_f \longrightarrow C_f/Y \cong \Sigma X$$

be the quotient map and

$$-\Sigma f : \Sigma X \longrightarrow \Sigma Y$$

be the map defined by

$$(-\Sigma f)(x \wedge t) = f(x) \wedge (1 - t).$$

Then the cofiber sequence generated by f *is*

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma \pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$

Remark 3.20. It is not very hard to show that this sequence is an exact sequence of pointed sets.

Proposition 3.21. For a based map $f : X \longrightarrow Y$ we have

$$\Sigma^n C_f \cong C_{\Sigma^n f} \qquad n > 1.$$

Proof. We show the case $n = 1$ and the general case follows by induction on n. Now consider the map

$$\Sigma C_f = \Sigma(Y \cup_f CX) \longrightarrow C_{\Sigma f} = \Sigma Y \cup_{\Sigma f} C\Sigma X$$

$$(c \wedge t) \longmapsto \begin{cases} c \wedge t & \text{if } c \in Y \\ (x \wedge t, s) & \text{if } c = (x, s) \in CX \end{cases}$$

One can easily show that this map is a homeomorphism. $\square$

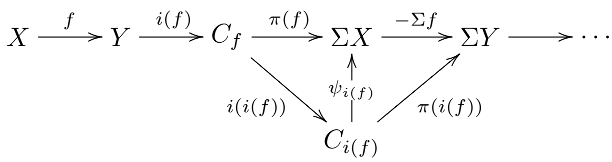
Proposition 3.22. *If $i : A \longrightarrow X$ is a cofibration, then quotient map*

$$\psi_i : C_i \longrightarrow C_i/CA \cong X/A$$

is a based homotopy equivalence.

We shall note the homotopy inverse of ψ_i, by ϕ_i. There is yet another important proposition that we use to show one of the main results of this section.

Proposition 3.23. *In the following diagram, the left triangle commutes and the right triangle commutes up to homotopy equivalence*



Theorem 3.24. *Suppose that X, Y, Z are three based spaces. The cofiber sequence generated by the map $f : X \longrightarrow Y$ induces the sequence*

$$\cdots \longrightarrow [\Sigma C_f, Z] \longrightarrow [\Sigma Y, Z] \longrightarrow [\Sigma X, Z] \longrightarrow [C_f, Z] \longrightarrow [Y, Z] \longrightarrow [X, Z]$$

which is an exact sequence of pointed sets, or of groups to the left of $[\Sigma X, Z]$, or of Abelian groups to the left of $[\Sigma^2 X, Z]$.

Proof. According to Remark 3.20 we have only to show that at each stage the inverse image of the basepoint is contained in the image of previous map. We now show a key point of the proof. For each pair of maps $f : X \longrightarrow Y, i(f) : Y \longrightarrow C_f$ where i is the inclusion of the codomain of f into its homotopy cofiber, and each map $g : Y \longrightarrow Z$ whose composition with f is null homotopic, there exists a map $\tilde{g} : C_f \longrightarrow Z$ whose restriction to Y is g. The sequence in question is thus exact at the first stage, but we shall show that each pair of consecutive maps in the sequence is of this form up to homotopy. To see this, consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C_f \\ & & {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \tilde{g}=g\cup h} & \\ & & Z & & \end{array}$$

where $h : g \circ f \simeq c_*$ is a based homotopy. Since h is constant on the set $X \times \{1\} \cup \{*\} \times I$ we may consider it as a map $CX \longrightarrow Z$, using the fact that $C_f = Y \cup_f CX, h$ and g induce the required map $\tilde{g}$. We can show by induction on k that there is a commutative diagram up to homotopy

$$\begin{array}{ccccccc} \Sigma^k X & \xrightarrow{\Sigma^k f} & \Sigma^k Y & \xrightarrow{\Sigma^k i(f)} & \Sigma^k C_f & \xrightarrow{\Sigma^k \pi(f)} & \Sigma^{k+1} X \\ \downarrow = & & \downarrow = & & \downarrow \simeq & & \downarrow = \\ \Sigma^k X & \xrightarrow[\Sigma^k f]{} & \Sigma^k Y & \xrightarrow[i(\Sigma^k f)]{} & C(\Sigma^k f) & \xrightarrow[\pi(\Sigma^k f)]{} & \Sigma^{k+1} X. \end{array}$$

Thus it suffices to show the exactness for the two pairs of maps $(i(f), \pi(f))$ and $(\pi(f), -\Sigma f)$. So let $[g] \in [C_f, Z]$ such that $g \circ i(f) \simeq c_*$. From the first argument we have that the sequence

$$Y \xrightarrow{i(f)} C_f \xrightarrow{i(i(f))} C_{i(f)}$$

induces the exact sequence

$$[C_{i(f)}, Z] \longrightarrow [C_f, Z] \longrightarrow [Y, Z]$$

so there exists $h : C_i(f) \longrightarrow Z$ such that $h \circ i(i(f)) \simeq g$. Applying Proposition 3.22 and Remark 3.16 we have $h \circ \phi_{i(f)} : \Sigma X \longrightarrow Z$ and applying the commutativity of the first triangle of the diagram of Proposition 3.23 $h \circ \phi_{i(f)} \circ \pi(f) \simeq g$ since $\psi_{i(f)} \circ i(i(f)) = \pi(f)$ implies that

$$i(i(f)) = \mathrm{id}_{\Sigma X} \circ i(i(f)) \simeq \phi_{i(f)} \circ \psi_{i(f)} \circ i(i(f)) = \phi_{i(f)} \circ \pi(f).$$

Thus the sequence

$$[\Sigma X, Z] \longrightarrow [C_f, Z] \longrightarrow [Y, Z]$$

is exact. To show the exactness of the sequence

$$[\Sigma Y, Z] \longrightarrow [\Sigma X, Z] \longrightarrow [C_f, Z]$$

suppose given $g : \Sigma Y \longrightarrow Z$ with $g \circ \pi(f) \simeq c_*$ so we have $g \circ \psi_{i(f)} : C_{i(f)} \longrightarrow Z$ Using the result of the first pair for the pair $(i(i(f)), \pi(f))$, there exists $h : \Sigma Y \longrightarrow Z$ such that $g \circ \psi_{i(f)} \simeq h \circ \pi(i(f))$ so

$$g = g \circ \mathrm{id}_{\Sigma X} \simeq g \circ \psi_{i(f)} \circ \phi_{i(f)} \simeq h \circ \pi(i(f)) \circ \phi_{i(f)}$$

and using the commutativity up to homotopy of the second triangle of the diagram of Proposition 3.23 we have $\pi(i(f)) \circ \phi_{i(f)} \simeq -\Sigma f$ since $\psi_{i(f)}$ and $\phi_{i(f)}$ are homotopic inverses of each other and therefore $g \simeq h \circ (-\Sigma f)$. □

Now we turn to the case of fibrations and the homotopy exact sequences associated to them.

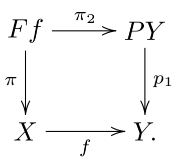
Definition 3.25. *A* based fibration *is a fibration where all maps and spaces in the definition of fibration are required to be based. We call them simply fibrations in this section.*

Remark 3.26. A based fibration is necessarily a fibration in the unbased definition. To see this we restrict to spaces of the form Y_+ in the test diagrams and observe that $Y_+ \wedge I_+ \cong (Y \times I)_+$. It is less obvious that if a map $p : E \longrightarrow B$ is a fibration in the unbased sense then it satisfies the covering homotopy property for test diagrams in which Y is nondegenerately based.

Remark 3.27. A based homotopy $X \wedge I_+ \longrightarrow Y$ is "the same thing as" a based map $X \longrightarrow F(I_+, Y)$ where $F(I_+, Y)$ is the space Y^I with the basepoint $c_* : I \longrightarrow Y$,the constant path at the basepoint of Y.

Definition 3.28. *In this section the* mapping path space Nf *of a map* $f : X \longrightarrow Y$ *is the subspace of* $X \times Y^I$*, defined by* $Nf = \{(x, \beta) \mid \beta(1) = f(x)\}$ *given the basepoint* $(*, c_*)$.

Definition 3.29. *The* homotopy fiber *of a map* $f : X \longrightarrow Y$*,* Ff *is the pullback of the following diagram*



Explicitly we have

$$Ff = X \times_f PY = \{(x, \beta) \mid \beta(1) = f(x)\} \subset X \times PY.$$

Remark 3.30. Since $\pi : Ff \longrightarrow X$ is a pullback of the fibration $p_1 : PY \longrightarrow Y$, it is a fibration.

Definition 3.31. *Let* $\iota : \Omega Y \longrightarrow Ff$ *be the inclusion* $\iota(\beta) = (*, \beta)$*. The following sequence is called the* fiber sequence generated by f

$$\cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega F f \xrightarrow{-\Omega\pi} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} Ff \xrightarrow{\pi} X \xrightarrow{f} Y$$

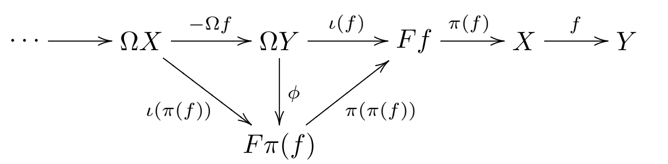
where

$$(-\Omega f)(\zeta)(t) = (f \circ \zeta)(1-t) \quad \text{for} \quad \zeta \in \Omega X.$$

Dual to the long exact sequences of pointed set induced from the cofiber sequences, we have the following theorem whose proof is dual to that of Theorem 3.24. But similarly to the proof of Theorem 3.24 we need two propositions before.

Proposition 3.32. *If* $p : E \longrightarrow B$ *is a fibration, then the inclusion* $\phi : p^{-1}(*) \longrightarrow Fp$ *defined by* $\phi(e) = (e, c_*)$ *is a based homotopy equivalence.*

Proposition 3.33. *In the following diagram, the right triangle commutes and the left triangle commutes up to homotopy*



Theorem 3.34. *Suppose that* X, Y *and* Z *are three based spaces. The fiber sequnece generated by the map* $f : X \longrightarrow Y$ *induces the sequnece*

$$\cdots \longrightarrow [Z, \Omega Ff] \longrightarrow [Z, \Omega X] \longrightarrow [Z, \Omega Y] \longrightarrow [Z, Ff] \longrightarrow [Z, X] \longrightarrow [Z, Y]$$

which is an exact sequence of pointed sets, or of groups to the left of $[Z, \Omega Y]$*, or of Abelian groups to the left of* $[Z, \Omega^2 Y]$*.*

Remark 3.35. The group structure of the space $[Z, \Omega Y]$ is imposed by Proposition 3.8 and Proposition 3.9 .

We have seen that for any space X and any map $f : X \longrightarrow Y$ between spaces we can define two other spaces and maps, namely $\Sigma X, \Omega X$ and $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ and $\Omega f : \Omega X \longrightarrow \Omega Y$. These constructions preserve identities and composition of maps and thus are functors from the category of pointed topological spaces to itself. Proposition 3.8 suggests that the pair (Σ, Ω) forms an adjoint pair. In fact it is the case, and naturality is straightforward. There exists *unit* and *counit* natural transformation for these functors, which are defined as follows.

Definition 3.36. *For a based space X and a based map $f : X \longrightarrow Y$ define*

$$\eta : X \longrightarrow \Omega\Sigma X \qquad \text{and} \qquad \varepsilon : \Sigma\Omega X \longrightarrow X$$

by $\eta(x)(t) = x \wedge t$ and $\varepsilon(\beta \wedge t) = \beta(t)$ for $x \in X, \in \Omega X$ and $t \in S^1$. And define

$$\eta : Ff \longrightarrow \Omega C_f \qquad \text{and} \qquad \varepsilon : \Sigma Ff \longrightarrow C_f$$

by

$$\eta(x, \gamma)(t) = \varepsilon(x, \gamma, t) = \begin{cases} \gamma(2t) & \text{if} \quad t \leq 1/2 \\ (x, 2t-1) & \text{if} \quad t \geq 1/2 \end{cases}$$

for $(x, \gamma) \in Ff$.

There is a connection between cofiber sequences and fiber sequences that is useful in the calculation of homotopy groups of some spaces, as we will see later.

Proposition 3.37. *Let $f : X \longrightarrow Y$ be a map between spaces. The following diagram is homotopy commutative where the top row is obtained by applying the functor to a part of the fiber sequence generated by f and the bottom row is obtained by applying the functor Ω to a part of the cofiber sequence generated by f:*

$$\begin{array}{ccccccccccc}
 & & \Sigma\Omega Ff & \xrightarrow{\Sigma\Omega p} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma\iota} & \Sigma Ff & \xrightarrow{\Sigma p} & \Sigma X \\
 & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \| \\
\Omega Y & \xrightarrow{\iota} & Ff & \xrightarrow{p} & X & \xrightarrow{f} & Y & \xrightarrow{i} & C_f & \xrightarrow{\pi} & \Sigma X \\
\| & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \\
\Omega Y & \xrightarrow[\Omega i]{} & \Omega C_f & \xrightarrow[\Omega\pi]{} & \Omega\Sigma X & \xrightarrow[\Omega\Sigma f]{} & \Omega\Sigma Y & \xrightarrow[\Omega\Sigma i]{} & \Omega\Sigma C_f. & &
\end{array}$$

Proof. If we number the squares of this diagram from the top left to the right bottom from 1 to 8 the commutativity of squares $1, 2, 7, 8$ follows from the naturality of η and ε. The commutativity of squares 3 and 4 is equivalent to the commutativity of squares 6 and 5 respectively, since the functors Σ and Ω are adjoint to eachother. Thus it suffices to see that squares 5 and 6 are homotopy commutative, which is a mechanical task. $\square$

Proposition 3.38. *Let $f : X \longrightarrow Y$ be a map of based spaces. Then the following diagram is homotopy commutative, where $j : X \longrightarrow M_f$ is the inclusion and $r : M_f \longrightarrow Y$ is the retraction and π is induced by the quotient map $M_f \longrightarrow C_f$*

$$\begin{array}{ccc} Fj = X \times_j PM_f & \xrightarrow{Fr = id \times Pr} & X \times_f PY = Ff \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \eta} & \\ & \Omega C_f & \end{array}$$

PROOF. We write $\beta \in PM_f$ as (β_1, β_2) and define a homotopy

$$H : X \times_j PM_f \times I \longrightarrow \Omega C_f$$

as follows

$$H(x, \beta, t)(s) = \begin{cases} \beta_1(\frac{2s}{2-t}) \wedge (1-t)\beta_2(\frac{2s}{2-t}) & \text{if} \quad 0 \leq s \leq \frac{2-t}{2} \\ x \wedge (2s-1) & \text{if} \quad \frac{2-t}{2} \leq s \leq 1. \end{cases}$$

It is now easy to verify that this map is indeed the required homotopy. $\square$

4 Homotopy Groups

In this section we define and describe one of the most important objects of algebraic topology, homotopy group of a topological space, these constructions make it possible to translate topological properties of a space in an algebraic language,in a homotopy-invariant fashion, indeed we want to study the homotopy type of a space so that these homotopy group functors must factor through the homootpy category of topological spaces. These invariants supply us with means to do algebraic manipulations which correspond to homotopy-invariant constructions in the category of topological spaces. Throughout this section we suppose some some familiarity we elementary algebraic topology such as fundamental groups and covering spaces.

Proposition 4.1. *We have the following canonical homeomorphisms*

$$S^n \cong S^1 \wedge \cdots \wedge S^1 \cong \Sigma S^{n-1} \cong \Sigma^n S^1 \cong I^n/\partial I^n \quad \textit{for} \quad n \geq 1$$

Definition 4.2. *For $n \geq 0$ and a based space X define the n-th* homotopy group *of X at the basepoint $*$ to be the set of homotopy classes of based maps $S^n \longrightarrow X$, i.e.*

$$\pi_n(X) = \pi_n(X, *) = [S^n, X].$$

Remark 4.3. In view of Proposition 3.9 $\pi_n(X)$ is a group for $n \geq 1$ and is an Abelian group for $n \geq 2$.

Proposition 4.4. *For $n \geq 0$ we have*

$$\pi_n(X) = \pi_{n-1}(\Omega X) = \cdots = \pi_0(\Omega^n X).$$

Proof. We show by induction on n that $\Omega^n X = F(S^n, X)$, indeed we have $\Omega X = F(S^1, X)$ by definition and from Proposition 3.8 and Proposition 4.1 we draw that

$$\Omega^n(X) = \Omega^{n-1}(\Omega X) = F(S^{n-1}, \Omega X) = F(\Sigma S^{n-1}, X) = F(S^n, X)$$

applying π_0 to both side of the equality, we have that $\pi_n(X) = \pi_0(\Omega^n X)$. Now replacing X by $\Omega^k X$ and n by $n-k$, one shows the required equalities. □

Proposition 4.5. *For $* \in A \subset X$, the homotopy fiber of the inclusion $i : A \longrightarrow X, Fi$ may be identified with the space of paths in X that begin at the basepoint and end in A. We note this space by$P(X; *, A)$.*

Definition 4.6. *For $n \geq 1$ define*

$$\pi_n(X, A) = \pi_n(X, A, *) = \pi_n(P(X; *, A)).$$

These are called relative homotopy groups*.*

Remark 4.7. Again we have that $\pi_n(X, A)$ is a group for $n \geq 2$ and an Abelian group for $n \geq 3$ and $\pi_n(X, A) = \pi_0(\Omega^{n-1} P(X; *, A))$.

Proposition 4.8. *For $n \geq 1$ we have*

$$\pi_n(X, A, *) = [(I^n, \partial I^n, J^n), (X, A, *)]$$

where

$$J^n = \partial I^{n-1} \times I \cup I^{n-1} \times \{0\} \subset I^n \quad \text{and} \quad J^1 = \{0\} \subset I$$

and we consider the homotopy classes of maps of triples,i.e. the image of each component is included in the corresponding component.

In the identification $Fi = P(X, *, A)$, the projection on A from Fi corresponds to the endpoint projection of $P(X, *, A)$ on A ,i.e., the endpoints of the paths in $P(X, *, A)$, which we note $p_1 : Fi \longrightarrow A$. Thus we have the fiber sequence generated by the inclusion $i : A \longrightarrow X$ as follows

$$\cdots \longrightarrow \Omega^2 A \longrightarrow \Omega^2 X \longrightarrow \Omega Fi \longrightarrow \Omega A \longrightarrow \Omega X \xrightarrow{\iota} Fi \xrightarrow{p_1} A \xrightarrow{i} X.$$

Applying $\pi_0(-) = [S^0, -]$ to this sequence and using Proposition 4.4, we obtain the long exact sequence associated to i which is

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(A) \longrightarrow \pi_0(X).$$

Remark 4.9. We can give explicitly the definition of $\partial : \pi_n(X, A) \longrightarrow \pi_{n-1}(A)$ as follows. By Proposition 4.8 we know that an element f of $\pi_n(X, A)$ is a map $(I^n, \partial I^n, J^n) \longrightarrow (X, A, *)$, thus restricting it to $(I^{n-1} \times \{1\}, \partial I^{n-1} \times \{1\})$ and using the fact that $S^{n-1} \cong I^{n-1}/\partial I^{n-1}$ we have a based map $S^{n-1} \longrightarrow A$ this map is the image of f under ∂. The maps $\pi_n(A) \longrightarrow \pi_n(X)$ and $\pi_n(X) \longrightarrow \pi_n(X, A)$ are induced by the inclusions $(A, *) \subset (X, *)$ and $(X, *, *) \subset (X, A, *)$.

Proposition 4.10. *Let $p : E \longrightarrow B$ be a fibration with B path connected. Fix a basepoint $* \in B$ and let F be the fibre over $*$ and fix a basepoint $* \in F \subset E$. Let $\phi : F \longrightarrow Fp$ be the based homotopy equivalence defined by Proposition 3.32 . Then the following diagram is homotopy commutative:*

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & \Omega^2 E & \xrightarrow{-\Omega\iota} & \Omega Fi & \xrightarrow{-\Omega p_1} & \Omega F & \xrightarrow{-\Omega i} & \Omega E & \xrightarrow{\iota} & Fi & \xrightarrow{p_1} & F & \xrightarrow{i} & E \\
& \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle -\Omega p} & & \downarrow{\scriptstyle \Omega\phi} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle -p} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} \\
\cdots \longrightarrow & \Omega^2 E & \xrightarrow[\Omega^2 p]{} & \Omega^2 B & \xrightarrow[-\Omega\iota]{} & \Omega Fp & \xrightarrow[-\Omega\pi]{} & \Omega E & \xrightarrow[-\Omega p]{} & \Omega B & \xrightarrow[\iota]{} & Fp & \xrightarrow[\pi]{} & E
\end{array}$$

where $Fi = P(E, *, F)$ and $p(\xi) = p \circ \xi \in \Omega B$ for $\xi \in Fi$.

Proof. The most right square and the third square are commutative which can be shown by writing down the maps explicitly. The next to last square is homotopy commutative, for define a homotopy $h : \iota \circ (-p) \simeq \phi \circ p_1$ by

$$h(\xi, t) = (\xi(t), p(\xi[1, t])) \quad \text{where} \quad \xi[1, t](s) = \xi(1 - s + st)$$

i.e., $\xi[1, t]$ is the path going from $\xi(1)$ to $\xi(t)$. The other squares are obtained from these three squares by applying the functor Ω,hence they are commutative as well. □

Proposition 4.11. *With same hypothesis and notations of last proposition we have that* $p_* : \pi_n(E, F) \longrightarrow \pi_n(B)$ *is an isomorphism for* $n \geq 1$.

Proof. Passing to long exact sequences of homotopy groups described above ,using the fact that homotopy equivalences induce isomorphisms on the homotopy groups and using the five lemma, we achieve the proof. □

Remark 4.12. This result could also be derived directly from the covering homotopy property of p.

Remark 4.13.

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \{*\}.$$

A little path lifting argument shows that $\pi_0(F) \longrightarrow \pi_0(E)$ *is a surjection.* Using ϕ_* to identify $\pi_*(F)$ with $\pi_*(Fp)$ where π_* stands for generic homotopy groups, we may rewrite the long exact sequence obtained from the bottom row of the diagram as

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \{*\}.$$

A little path lifting argument shows that $\pi_0(F) \longrightarrow \pi_0(E)$ is a surjection.

Now we see some examples of homotopy groups.

Example 4.14. If X is contractible, then $\pi_n(X) = 0$ for all $n \geq 0$. Since X is homotopic to singleton whose homotopy groups are trivial.

Example 4.15. If X is discrete then $\pi_n(X) = 0$ for all $n \geq 1$. Since fixing a basepoint in X, and using the fact that S^n is connected for $n \geq 1$ we observe that any based and continuous map $S^n \longrightarrow X$ has image in $\{*\}$ thus it is trivial.

Example 4.16. If $p : E \longrightarrow B$ is a covering, then $p_* : \pi_*(E) \longrightarrow \pi_*(B)$ is an isomorphism for all $n \geq 2$. To see this, consider the exact sequence of Remark 4.13 and observe that for a covering the preimage of a point is a discrete set, thus $\pi_{n-1}(F) = 0$ for $n \geq 1$ from the previous example so $\pi_n(E) \cong \pi_n(B)$.

Example 4.17. $\pi_1(S^1) = \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n \geq 2$. We already know the first statement . To see the second statement observe that the map $p : \mathbb{R} \longrightarrow S^1$,$t \longmapsto e^{i\pi t}$ is a covering, so we have $\pi_n(S^1) \cong \pi_n(\mathbb{R})$ which is trivial since $\mathbb{R}$ is a contractible space.

Example 4.18. If $i \geq 2$, then $\pi_1(\mathbb{R}P^i) = \mathbb{Z}_2$ and $\pi_n(\mathbb{R}P^i) \cong \pi_n(S^i)$ for $n \geq 2$. Again the first statement is a covering space argument,for the second one observe that the projection

$$\pi : S^i \longrightarrow \mathbb{R}P^i = S^i/\mathbb{Z}_2$$

is a covering, and use Example 4.16 .

Proposition 4.19. *For all spaces X and Y and all n, we have*

$$\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y).$$

Proof. Projections $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ define a canonic function that sends a given map $f : S^n \longrightarrow X \times Y \in \pi_n(X \times Y)$ to the pair $(p_1 \circ f, p_2 \circ f) \in \pi_n(X) \oplus \pi_n(Y)$. Conversely given a pair of maps $(f_1, f_2) \in \pi_n(X) \oplus \pi_n(Y)$ we can define a map

$$f : S^n \longrightarrow X \times Y \in \pi_n(X \times Y)\ ,\ \ s \longmapsto (f_1(s), f_2(s))$$

. It is now easy to verify that these functions are homomorphisms, one inverse of the other. □

Proposition 4.20. *If $i < n$, then $\pi_i(S^n) = 0$.*

Proposition 4.21. *$\pi_2(S^2) \cong \mathbb{Z}$ and $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \geq 3$.*

Proposition 4.22. *If X is the colimit of a sequence of inclusions $X_i \longrightarrow X_{i+1}$ of based spaces, then the natural map*

$$\text{colimit}_i\ \pi_n(X_i) \longrightarrow \pi_n(X)$$

is an isomorphism for each n.

So far we have fixed a basepoint and considered homotopy groups with respect to it,now we want to know the relation between homotopy groups when we change the basepoint. Since the inclusion of the basepoint in S^n is a cofibration, we conclude from Proposition 2.10 that the evaluation at the basepoint $p : X^{S^n} \longrightarrow X$ is a fibration. We can identify $\pi_n(X, x)$ with $\pi_0(F_x)$,where F_x is the fiber over $x \in X$ which is in fact the loop space $\Omega^n X$ with respect to the base point x,since two different points of F_x lie in the same path component if and only if there is a based homotopy $h : S^n \times I \longrightarrow X$, and so homotopic maps $S^n \longrightarrow X \in \pi_n(X, x)$ correspond to points in the same path components of $\pi_0(F_x)$. We have seen in Proposition 2.16 that a path class $[\xi] : I \longrightarrow X$ from

x to x' induces a homotopy equivalence $\tau[\xi] : F_x \longrightarrow F_{x'}$, we continue to write $\tau[\xi]$ for the induced bijection

$$\tau[\xi] : \pi_n(X, x) \longrightarrow \pi_n(X, x').$$

Definition 4.23. *For any space X, the* folding map $\nabla : X \vee X \longrightarrow X$ *is the unique map $X \vee X \longrightarrow X$ which restricts to the identity map $X \longrightarrow X$ on each wedge summand.*

Definition 4.24. *The* pinch map $\mathfrak{p} : S^n \longrightarrow S^n \vee S^n$ *is the map that is obtained by collapsing an equator to the basepoint.*

Proposition 4.25. *Given two maps $f, g : S^n \longrightarrow X$, then $[f] + [g]$ is the homotopy class of the composite*

$$S^n \xrightarrow{\mathfrak{p}} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X.$$

Remark 4.26. There is another way to see the addition in $\pi_n(X, x)$. The cofibration $* \longrightarrow S^n$ induces the following pushout diagram

$$\begin{array}{ccc} * & \longrightarrow & S^n \\ \downarrow & & \downarrow \\ S^n & \longrightarrow & S^n \vee S^n \end{array}$$

thus the inclusion $S^n \longrightarrow S^n \vee S^n$ and its composition with $* \longrightarrow S^n$ are cofibrations, by Proposition 1.5. Again from Proposition 2.10 the induced map $X^{S^n \vee S^n} \longrightarrow X$ is a fibration and we have a diagram

$$\begin{array}{ccc} X^{S^n \vee S^n} & \xrightarrow{p^{\mathfrak{p}}} & X^{S^n} \\ \downarrow & & \downarrow p \\ X & = & X. \end{array}$$

$$S^n \xrightarrow{\mathfrak{p}} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

by definition. Thus the addition in $\pi_n(X, x)$ is obtained by restricting the map $p^{\mathfrak{p}}$ to the fiber over x and then applying π_0.

Proposition 4.27. *The bijection $\tau[\xi] : \pi_n(X, x) \longrightarrow \pi_n(X, x')$ is an isomorphism.*

PROOF. Using Proposition 2.18 which states the naturality of translations of fibers with respect to maps of fibrations, and with a little extra argument we can show that the following diagram is homotopy commutative*the fiber over x in left hand fibration is the product $F_x \times F_x$,where F_x is the fiber over x in the right hand fibration, since from the last pushout diagram any map $S^n \vee S^n \longrightarrow X$ is a pair of maps $S^n \longrightarrow X$ which coincide on the basepoint of S^n. Thus by identification $F_x = \Omega^n X$, the fiber over x in the left hand fibration is $\Omega^n X \times \Omega^n X$. Now given two elements $f, g \in \Omega X$,we see the pair (f, g) as an element of the fiber in $X^{S^n \vee S^n}$ which is sent into the fiber $F_x = \Omega^n X$ by the map $p^{\mathfrak{p}}$,but this element is exactly the composite*

$$S^n \xrightarrow{\mathfrak{p}} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

by definition. Thus the addition in $\pi_n(X, x)$ is obtained by restricting the map $p^{\mathfrak{p}}$ to the fiber over x and then applying π_0.

Proposition 4.28. *The bijection $\tau[\xi] : \pi_n(X, x) \longrightarrow \pi_n(X, x')$ is an isomorphism.*

PROOF. Using Proposition 2.18 which states the naturality of translations of fibers with respect to maps of fibrations, and with a little extra argument we can show that the following diagram is homotopy commutative

$$\begin{array}{ccc} F_x \times F_x & \longrightarrow & F_x \\ {\scriptstyle \tau[\xi]\times\tau[\xi]}\downarrow & & \downarrow{\scriptstyle \tau[\xi]} \\ F_{x'} \times F_{x'} & \longrightarrow & F_{x'}. \end{array}$$

Addition in homotopy groups are induced by the horizontal maps on passage to π_0, hence these translations of fibers are compatible with additions in homotopy groups, and they are homomorphisms. □

There is a similar result for relative homotopy groups,the idea is the same, for $a \in A$ we identify the group $\pi_n(X, A, a)$ with the homotopy class of maps of triples $[(CS^{n-1}, S^{n-1}, *), (X, A, *)]$, via the homotopy equivalence

$(I^n, \partial I^n, J^n) \cong (CS^{n-1}, S^{n-1}, *)$ obtained by quotienting out J^n. We have two cofibrations $\{*\} \longrightarrow S^{n-1}$ and $S^{n-1} \longrightarrow CS^{n-1}$ which give rise to the fibration

$$p : (X, A)^{(CS^{n-1}, S^{n-1})} \longrightarrow A$$

that to each function $(CS^{n-1}, S^{n-1}) \longrightarrow (X, A)$ associate its evaluation at the basepoint. Again we identify $\pi_n(X, A)$ with $\pi_0(F_a)$. A path class $[\alpha] : I \longrightarrow A$

from a to a' induces a homotopy equivalence $\tau[\alpha] : F_{aa'}$,and again we write $\tau[\alpha]$ for the induced isomorphism

$$\tau[\alpha] : \pi_n(X, A, a) \longrightarrow \pi_n(X, A, a').$$

From the naturality stated in Proposition 2.18 we deduce the following theorem which shows how the homotopy groups behave when we change the basepoint. The proof uses essentially Proposition 2.18 .

Theorem 4.29. *If $f : (X, A) \longrightarrow (Y, B)$ is map of pairs and $\alpha : I \longrightarrow A$ is a path from a to a', then the following diagram commutes*

$$\begin{CD}
\pi_n(X, A, a) @>{f_*}>> \pi_n(Y, B, f(a)) \\
@V{\tau[\alpha]}VV @VV{\tau[f\circ\alpha]}V \\
\pi_n(X, A, a') @>{f_*}>> \pi_n(Y, B, f(a')).
\end{CD}$$

If $h : f \simeq f'$ is a homotopy of maps of pairs and $h(a)(t) = h(a, t)$, then the following diagram commutes

$$\begin{array}{ccc}
 & \pi_n(X, A, a) & \\
 {}^{f_*}\swarrow & & \searrow^{f'_*} \\
\pi_n(Y, B, f(a)) & \xrightarrow[\tau[h(a)]]{} & \pi_n(Y, B, f'(a)).
\end{array}$$

As a consequence we have the following proposition.

Proposition 4.30. *If $f : X \longrightarrow Y$ is map between topological spaces, and $\alpha : I \longrightarrow A$ a path from a to a' then the following diagram commutes*

$$\begin{CD}
\pi_n(X, a) @>{f_*}>> \pi_n(Y, f(a)) \\
@V{\tau[\alpha]}VV @VV{\tau[f\circ\alpha]}V \\
\pi_n(X, a') @>{f_*}>> \pi_n(Y, f(a')).
\end{CD}$$

If $h : f \simeq f'$ is a homotopy and $h(a)(t) = h(a, t)$, then the following diagram commutes

$$
\begin{array}{ccc}
 & \pi_n(X,a) & \\
 {}^{f_*}\swarrow & & \searrow^{f'_*} \\
\pi_n(Y,f(a)) & \xrightarrow[\tau[h(a)]]{} & \pi_n(Y,f'(a)).
\end{array}
$$

Proposition 4.31. *A homotopy equivalence of spaces or of pairs of spaces induces an isomorphism on all homotopy groups.*

Proof. We show only the claim for the relative case, other one is the same. So, suppose that we have a pair of maps of pairs $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (X, A)$ and a pair of homotopy of pairs $h : f \circ g \simeq id_Y$ and $k : g \circ f \simeq id_X$, we show that

$$f_* : \pi_n(X, A, a) \longrightarrow \pi_n(Y, B, f(a))$$

is an isomorphism. We now that the induced homomorphism on homotopy groups by the identity map is the identity homomorphism,and that the composition of induced homomorphisms of two maps is the induced homomorphism of the composition of them, now using the second diagram of the last theorem we have

$$
\begin{array}{ccc}
 & \pi_n(X,A,a) & \\
 {}^{id_{X*}}\swarrow & & \searrow^{g\circ f_* = g_*\circ f_*} \\
\pi_n(X,A,a) & \xrightarrow[\tau[k(a)]]{} & \pi_n(X,A,g\circ f(a)).
\end{array}
$$

Since $\tau[k(a)]$ is an isomorphism, it follows that

$$f_* : \pi_n(X, A, a) \longrightarrow \pi_n(Y, B, f(a))$$

is a monomorphism and that

$$g_* : \pi_n(Y, B, f(a)) \longrightarrow \pi_n(X, A, g \circ f(a))$$

is an epimorphism. Similarly, we have

$$g_* : \pi_n(Y, B, f(a)) \longrightarrow \pi_n(X, A, g \circ f(a))$$

is a monomorphism and that

$$f_* : \pi_n(X, A, g \circ f(a)) \longrightarrow \pi_n(Y, B, f \circ g \circ f(a))$$

is an epimorphism. Now the statement of the Theorem 4.28 says that for the path $k_a = k(a, -) : I \longrightarrow X$ we have a commutative diagram

$$\begin{array}{ccc}
\pi_n(X,A,a) & \xrightarrow{f_*} & \pi_n(Y,B,f(a)) \\
\Big\downarrow{\scriptstyle \tau[k_a]} & & \Big\downarrow{\scriptstyle \tau[f\circ k_a]} \\
\pi_n(X,A,g\circ f(a)) & \xrightarrow{f_*} & \pi_n(Y,B,f\circ g\circ f(a)).
\end{array}$$

$\tau[k_a]$ and $\tau[f \circ k_a]$ are isomorphisms,

$$f_* : \pi_n(X, A, g \circ f(a)) \longrightarrow \pi_n(Y, B, f \circ g \circ f(a))$$

is surjective, thus the monomorphism

$$f_* : \pi_n(X, A, a) \longrightarrow \pi_n(Y, B, f(a))$$

is an isomorphism. □

Now we define one of the most important classes of maps in the category of topological spaces. The homotopy theory is in part the study of these maps. As we will see in the next section they are isomorphisms in the homotopy category of CW-complexes.

Definition 4.32. *A map $e : Y \longrightarrow Z$ is an n-*equivalence *if for all $y \in Y$, the induced map*

$$e_* : \pi_q(Y, y) \longrightarrow \pi_q(Z, e(y))$$

is an isomorphism for $q < n$ and a surjection for $q = n$. e is a weak equivalence *if it is an n-equivalence for all n or equivalently if the induced homomorphisms on all homotopy groups are isomorphisms.*

Remark 4.33. If in the definition the space Y is path-connected, then it suffices to verify the induced homomorphisms for a fixed basepoint.

Proposition 4.34. *Any homotopy equivalence is a weak equivalence.*

Proof. This is clear from proposition 4.30 . □

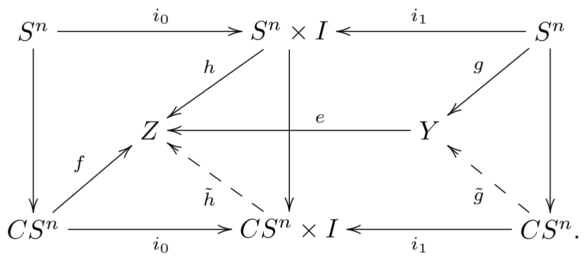
There is a powerful proposition which we shall use many times in the following sections. We take CX to be the unreduced cone. Again we view $\pi_{n+1}(X, x)$ as the set of relative homotopy classes of maps

$$(CS^n, S^n) \longrightarrow (X, x).$$

Proposition 4.35. *The following conditions on a map $e : Y \longrightarrow Z$ are equivalent.*

(i) *For any $y \in Y, e_* : \pi_q(Y, y) \longrightarrow \pi_q(Z, e(y))$ is an injection for $q = n$ and a surjection for $q = n + 1$.*

(ii) *Given maps* $f : CS^n \longrightarrow Z, g : S^n \longrightarrow Y$*, and* $h : S^n \times I \longrightarrow Z$ *such that* $f \mid S^n = h \circ i_0$ *and* $e \circ g = h \circ i_1$ *in the following diagram, there are maps* $\tilde{g}$ *and* $\tilde{h}$ *that make the entire diagram commute*



(iii) *The conclusion of* (ii) *holds when* $f \mid S^n = e \circ g$ *and* h *is the constant homotopy at this map, i.e.,* $h(s, t) = f(s) = e \circ g(s)$ *for all* $s \in S^n$ *and* $t \in I$*.*

Proof. (ii)$\Longrightarrow$ (iii). This is trivial.
(iii)$\Longrightarrow$(i). For $n = 0$ observe that a map $f : CS^0 \longrightarrow Z$ is the same thing as a path in Z from in $f(s_0, 0)$ to $f(s_1, 0)$. Thus if there is a path in Z from $e(y_o)$ to $e(y_1)$ we can see it as a map $f : CS^0 \longrightarrow Z$, the existence of $\tilde{g} : CS^0 \longrightarrow Y$ which makes the diagram commute, implies the existence of a path in Y from y_0 and y_1. Now suppose that $n > 0$, and that $e_*([g]) = 0$ that is $e \circ g \simeq c_{e(y)}$, where $c_{e(y)}$ is the constant map at $e(y)$. So there is a homotopy $f : S^n \times I \longrightarrow Z$ with $f_0 = e \circ g$ and $f_1 = c_{e(y)}$,we can see this homotopy as a map $f : CS^n \longrightarrow Z$ since it is constant on the set $X \times \{1\}$. Thus we have a map $f : CS^n \longrightarrow Z$ whose restriction to S^n is egAgain the lifting $\tilde{g}$ gives a homotopy $g \simeq c_y$ thus we have $[g] = 0$ in $\pi_n(Y, y)$,this shows the injectivity. Now we show that $\pi_{n+1}(e)$ is surjective. Given a map $f \in \pi_{n+1}(Z, e(y))$, as we said before we have $f : (CS^n, S^n) \longrightarrow (Z, e(y))$,now let g be the constant map at y, so we have a map $\tilde{g} : (CS^n, S^n) \longrightarrow (Y, y)$ and a homotopy $\tilde{h} : f \simeq e \circ \tilde{g}$ relative to S^n, so $\tilde{g} \in \pi_{n+1}(Y, y)$ is mapped to f. Hence the surjectivity of $\pi_{n+1}(e)$.
(i)$\Longrightarrow$(ii). We give a sketch of the proof. The idea is to use (i) to show that the nth homotopy group of the fiber F_y over y is trivial, and to use the given part of the diagram to construct $\tilde{g}$ and $\tilde{h}$. □

5 CW-Complexes

We define a large class of spaces, called CW-complexes, which play a very important role in homotopy theory and are the cornerstones of cellular homology theory. As we said before and we will see in this section any weak equivalence between them is a homotopy equivalence,and we will see that any space is "weakly" equivalent to a CW-complex.

Definition 5.1. *The unit n-disk $\mathbb{D}^{n+1}$ is the set $\{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$. The n-sphere $S^n \subset \mathbb{D}^{n+1}$ is the set $\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} = \partial\mathbb{D}^{n+1}$.*

Definition 5.2. *A* CW-complex *X is the colimit of the successive inclusions $X^n \longrightarrow X^{n+1}$ of subspaces X^n,that are constructed by the following inductive procedure. X^0 is a discrete set of points (called vertices),suppose that we have constructed X^k for $k \leq n$, and suppose that A_{n+1} is an indexing set and for any $\alpha \in A_{n+1}$ we have an* attaching map *$j_\alpha : S^n_\alpha \longrightarrow X^n$, then X^{n+1} is the pushout of the following diagram*

$$\begin{array}{ccc} \coprod_{\alpha \in A_{n+1}} S^n_\alpha & \xrightarrow{\coprod j_\alpha} & X^n \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in A_{n+1}} \mathbb{D}^{n+1}_\alpha & \xrightarrow[\coprod \tilde{j}_\alpha]{} & X^{n+1}. \end{array}$$

That is, X^{n+1} is constructed from X^n by attaching $(n+1)$-disks along attaching maps j_α. More explicitly X^{n+1} is the quotient space of $X^n \cup (\coprod_\alpha \mathbb{D}^{n+1})$ by the identification $j_\alpha(s) = s$, for all $\alpha \in A_{n+1}$ and $s \in S^n$. Each resulting map $i_\alpha : \mathbb{D}^{n+1}_\alpha \longrightarrow X^{n+1}$ or by abuse of language the image of such maps, is called an $(n+1)$-cell *or simply a* cell*. The space X^n is called the* n-skeleton*. If the 0-skeleton X^0 is the disjoint union of a space A with a (possibly empty) discrete set,then the resulting space is the* relative CW-complex *(X, A). In both relative or absolute case, we say that the CW-complex has dimension $\leq n$ if $X = X^n$. A CW-complex is finite if it has finitely many cells. A subcomplex Y of a CW-complex X is a subspace and a CW-complex such that for each $n \geq 0$ its indexing set B_n is contained in A_n and the maps $\mathbb{D}^n_\alpha \longrightarrow X^n$ are composition of the inclusion $Y \longrightarrow X$ and the maps $\mathbb{D}^n_\alpha \longrightarrow Y^n$ for all $\alpha \in B_n$. In other word, Y is the union of some of the cells of X. A map of pairs $f : (X, A) \longrightarrow (Y, B)$ between relative CW-complexes is said to be* cellular *if $f(X^n) \subseteq Y^n$ for all n.*

Remark 5.3. The maps $\mathbb{D}^n \longrightarrow X^n$ are sometimes called *characteristic maps*.

Remark 5.4. When A is a subcomplex of a CW-complex X we can view (X, A) as a relative CW-complex.

Of course the topology of the colimit is the weak topology, i.e.,a subspace is closed if and only if its intersections with skeleta are closed.

Proposition 5.5. *If X is a CW-complex, then each cell is contained in a finitely many X^n.*

Example 5.6. For any $n \geq$, the n-sphere S^n is a CW-complex with one vertex $\{*\}$ and one n-cell. The attaching map is the obvious map $S^{n-1} \longrightarrow \{*\}$, then the characteristic map $\mathbb{D}^n \longrightarrow S^n$ is the projection $\mathbb{D}^n \longrightarrow \mathbb{D}^n/S^{n-1} \cong S^n$. If $m < n$ then the only cellular map $S^m \longrightarrow S^n$ is the trivial map and if $m \geq n$ then every map $S^m \longrightarrow S^n$ is cellular.

Example 5.7. $\mathbb{R}P^n$ is a CW-complex with m-skeleton $\mathbb{R}P^m$ and with one m-cell for each $m \leq n$. The attaching map is the projection $j : S^{n-1} \longrightarrow S^{n-1}/\mathbb{Z}_2 \cong \mathbb{R}P^{n-1}$, thus $\mathbb{R}P^n$ is homeomorphic to $\mathbb{R}P^{n-1} \cup_j \mathbb{D}^n$. Explicitly write $\overline{x} = [x_1, ..., x_{n+1}], \sum x_i^2 = 1$, for a typical point of $\mathbb{R}P^n$. Then $\overline{x}$ is in $\mathbb{R}P^{n-1}$ if and only if $x_{n+1} = 0$, the required homeomorphism is obtained by identifying $\mathbb{D}^n$ and its boundary sphere with the upper hemisphere

$$\{(x_1, ..., x_{n+1}) \mid \sum x_i^2 = 1 \text{ and } \ x_{n+1} \geq 0\}$$

and its boundary.

The category of CW-complexes is closed under many constructions of topological spaces, so we can make complexes from ones given. The following propositions are not very hard to prove, they follow directly from the definitions.

Proposition 5.8. *If (X, A) is relative CW-complex, then the quotient space X/A is a CW-complex with a vertex corresponding to A and one n-cell for each relative n-cell of (X, A).*

Proposition 5.9. *For CW-complexes X_i with basepoints that are vertices, the wedge product $\bigvee_i X_i$, is a CW-complex which contains each X_i as a subcomplex. More over we have $(\bigvee_i X_i)^n = \bigvee_i X_i^n$.*

Definition 5.10. *The* Euler characteristic $\chi(X)$ *of a finite CW-complex is the alternating sum $\sum(-1)^n \gamma_n(X)$, where $\gamma_n(X)$ is the number of n-cells of X.*

Proposition 5.11. *Let A be a subcomplex of a CW-complex X, let Y be CW-complex, and let $f : A \longrightarrow Y$ be a cellular map, then the pushout $Y \cup_f X$ is a CW-complex with n-skeleton $Y^n \cup_f X^n$, and with Y as a subcomplex and has one cell for each cell of X that is not in A. The quotient complex $(Y \cup_f X)/Y$ is isomorphic to X/A. Further more if X and Y are finite we have the following formula relating the Euler characteristics $\chi(A), \chi(X), \chi(Y)$ and $\chi(Y \cup_f X)$*

$$\chi(Y \cup_f X) = \chi(Y) + \chi(X) - \chi(A).$$

Proof. Only the last statement about the Euler characteristics needs a verification, the others follow immediately from the definition. In fact we have $\gamma_n(Y \cup_f X) = \gamma_n(Y) + \gamma_n(X) - \gamma_n(A)$, since the map f is cellular and therefore the n-cells of A are sent to Y^n , from the first part of the proposition we have that $(Y \cup_f X)^n = Y^n \cup_f X^n$ thus the n-cells of A are have double contribution in the sum $\gamma_n(Y) + \gamma_n(X)$ once as n-cells of Y and once as n-cells of X, but this is superfluous, and we have to subtract once the number of n-cells of A, $\gamma_n(A)$ indeed we have $\gamma_n(Y \cup_f X) = \gamma_n(Y) + \gamma_n(X) - \gamma_n(A)$, which shows the desired result on Euler characteristic. □

Proposition 5.12. *The colimit of a sequence of inclusions of subcomplexes into CW-complexes, $X_i \longrightarrow X_{i+1}$, is a CW-complex that contains each of the X_i as a subcomplex.*

Proposition 5.13. *For p, q with $p+q = n$, there is a canonical homeomorphism*

$$(\mathbb{D}^n, S^{n-1}) \cong (\mathbb{D}^p \times \mathbb{D}^q, \mathbb{D}^p \times S^{q-1}, S^{p-1} \times \mathbb{D}^q).$$

Proof. This follows immediately from the homeomorphisms

$$\mathbb{D}^n \cong I^n \qquad \text{and} \qquad S^n \cong I^n/\partial I^n \qquad \text{and} \qquad I^n = I^p \times I^q.$$

□

This proposition allows us to endow the product of CW-complexes with a CW-structure. Explicitly we have the following result

Proposition 5.14. *The product $X \times Y$ of CW-complexes X and Y is a CW-complex with an n-cell for each pair consisting of a p-cell of X and a q-cell of Y, with $p + q = n$. In particular For a CW-complex X, the cylinder $X \times I$ is a CW-complex that contains $X \times partial I$ as a subcomplex and in addition, has one $(n + 1)$-cell for each n-cell of X.*

Proof. One can easy show by induction on n that the n-skeleton of $X \times Y$, is the space

$$(X \times Y)^n = \bigcup_{p+q=n} (X^p \times Y^q)$$

note also that the indexing set for the product complex in dimension n is $\coprod_{p+q=n} A_p \times B_q$, where A_p and B_q are respectively the indexing set of X and Y in dimensions p and q. For the last statement we use the fact that the unit interval I is a CW-complex with two vertices and one 1-cell attached to them in the obvious way, and the result follows from the general case. □

We can now produce many useful spaces from a given CW-complex, and to lie still in the category of CW-complexes.

Proposition 5.15. *Let X and Y be CW-complexes, X' and Y' be based CW-complexes with basepoints that are vertices, and let $f : X \longrightarrow Y$ be a cellular map, then the following spaces are all CW-complexes*

(i) *The smash product $X' \wedge Y'$.*

(ii) *The suspension $\Sigma X'$.*

(iii) *The cone CX'.*

(iv) *The mapping cylinder M_f.*

(v) *The mapping cone C_f.*

PROOF. To see that $X' \wedge Y'$ is a CW-complex, recall from Proposition 5.9 that the wedge product $X' \vee Y'$ is a CW-complex, and from Proposition 5.14 that the product $X' \times Y'$ is a CW-complex, now the space $X' \wedge Y' = X' \times Y'/X' \vee Y'$ is a CW-complex by Proposition 5.8, since the wedge $X' \vee Y'$

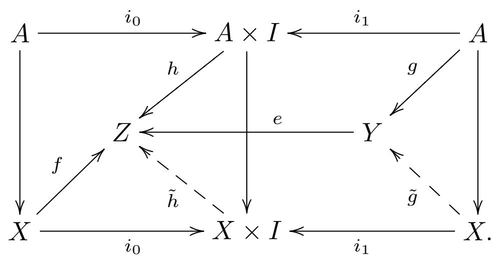
is a subcomplex of the product $X' \times Y'$, this shows (i). (ii) follows from (i), since $\Sigma X' = X' \wedge S^1$ and S^1 is a CW-complex from Example 5.6 . (iii) is clear from Proposition 5.14 and the definition of the cone CX'. To see (iv), we apply Proposition 5.11 to the relative CW-complex $(X \times I, X)$ and the map f . A same argument shows (v). □

In view of Proposition 5.14 we can define an appropriate homotopy in the category of CW-complexes,

Definition 5.16. *A* cellular homotopy $h : f \simeq f'$ *between cellular maps $X \longrightarrow Y$ of CW-complexes is a homotopy that is itself a cellular map $h : X \times I \longrightarrow Y$*

The last proposition in the last section, has a powerful consequence and generalization which allows to show many results about CW-complexes. This generalization is known as **HELP**(=homotopy extension and lifting property).

Theorem 5.17 (HELP). *Let (X, A) be a relative CW-complex of dimension $\leq n$ and let $e : Y \longrightarrow Z$ be an n-equivalence. Then given maps $f : X \longrightarrow Z, g : A \longrightarrow Y$ and $h : A \times I \longrightarrow Z$ such that $f \mid A = h \circ i_0$ and $e \circ g = h \circ i_1$ in the following diagram, there are maps $\tilde{g}$ and $\tilde{h}$ that makes the entire diagram commute*



PROOF. It is enough by induction on i to consider the case that $A = X^i$ and $X = X^{i+1}$. Then working one cell at a time it reduces to the case where $A = S^i$ and $X = \mathbb{D}^{i+1}$ which follows directly from Proposition 4.34 at the end of last section. □

Remark 5.18. This theorem says in part that if e is the identity map of Y, then the inclusion $A \longrightarrow X$ has the homotopy extension property, and therefore is a cofibration.

Remark 5.19. By passage to colimits, n can be ∞ in the theorem.

Theorem 5.20 (Whitehead). *If X is CW-complex and $e : Y \longrightarrow Z$ is an n-equivalence, then $e_* : [X, Y] \longrightarrow [X, Z]$is a bijection if $\dim(X) < n$ and a surjection if $\dim(X) = n$, where $[X, Y]$stands for the homotopy classes of unbased maps. In particular if e is weak equivalence then the result holds for any CW-complex X.*

PROOF. Given a map $f : X \longrightarrow Z$, an application of last theorem to the pair $(X, \emptyset)$, gives at the same time a map $g : Y \longrightarrow Z$ and a homotopy $h : X \longrightarrow Z$ satisfying $h_0 = f$ and $h_1 = e \circ g$, hence $[f] = [e \circ g]$ in $[X, Z]$, and therefore e_* is surjective. Given two maps $\phi_0, \phi_1 \in [X, Y]$ with $[e \circ \phi_0] = [e \circ \phi_1]$ in $[X, Z]$, so there is a homotopy $f : X \times I \longrightarrow Z$ with $f_i = \phi_i, i = 0, 1$, now apply last theorem to the pair $(X \times I, X \times \partial I)$, with $h : (X \times \partial I) \times I \longrightarrow Z$ the constant homotopy

$$h(x, t_i, s) = e \circ \phi_i(x), \quad i = 0, 1, \quad \text{and} \quad s \in I.$$

This gives the required homotopy between ϕ_0 and ϕ_1, thus e_* is injective. □

Theorem 5.21 (Whitehead). *An n-equivalence between CW-complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW-complexes is a homotopy equivalence.*

PROOF. Let $e : Y \longrightarrow Z$ satisfy either hypothesis, then from last theorem we have that $e_* : [Z, Y] \longrightarrow [Z, Z]$ is a bijection and so there is map $f : Z \longrightarrow Y$ such that $e \circ f \simeq id_Z$, thus $e \circ f \circ e \simeq e$ and since the map $e_* : [Y, Y] \longrightarrow [Y, Z]$ is also a bijection from last theorem, and the elements $[id_Y]$ and $[f \circ e]$ are mapped to the same element $[e]$ we deduce that $f \circ e \simeq$idY. □

Remark 5.22. This theorem says that an n-equivalence between CW-complexes of dimension less than n is a weak equivalence since any homotopy equivalence is a weak equivalence, so we have only to check bijectivity of the first $n-1$ homotopy groups to have the isomorphism on all homotopy groups

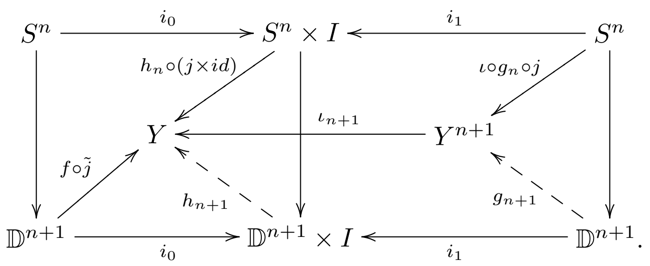
Whitehead theorem states that the homotopy groups are, in a sense, a complete homotopy invariants for a CW-complex. So it would be of great interest to be able to approximate a topological space with a CW-complex and to work in the homotopy category of CW-complexes in which the weak equivalences are inversible, rather than the larger category of topological spaces. Surprisingly, this approximation exists and is functorial as we will see, but before we prove a result, that says that any map between CW-complexes is homotopic to a cellular map. This result is very helpful, since the cellular maps behave much better that general maps.

Definition 5.23. *A space X is said to be n-*connected *if $\pi_q(X, x) = 0$ for $0 \leq q \leq n$ and all $x \in X$. A pair (X, A) is said to be n-*connected *if $\pi_0(A) \longrightarrow \pi_0(X)$ is surjective and $\pi_q(X, A, a) = 0$ for $1 \leq q \leq n$ and all $a \in A$. Equivalently the pair (X, A) is n-connected if the inclusion $A \longrightarrow X$ is an n-equivalence.*

Proposition 5.24. *A relative CW-complex (X, A) with no m-cells for $m \leq n$ is n-connected. In particular, (X, X^n) is n-connected for any CW-complex X.*

Theorem 5.25 (Cellular Approximation). *Any map $f : (X, A) \longrightarrow (Y, B)$ between relative CW-complexes is homotopic relative to A to a cellular map.*

PROOF. We construct this map inductively, that is we construct successively for all n a map $g_n : X^n \longrightarrow Y^n$ that is homotopic to the restriction of f to X^n. Since the CW-complex Y is constructed from Y^0 by attaching cells, any point of Y can be connected to a point in Y^0 by a path, now for each point of X^0 A pick a point in Y^0 that is connected to it via a path, this gives a map from $g_0 : X^0 \longrightarrow Y^0$, if we define the images of points in A as their images under f. This map is homotopic relative to A to $f \mid X^0$, this was the first step of the induction. Now suppose that we have constructed a map $g_n : X^n \longrightarrow Y^n$ and a homotopy $h_n : X^n \times I \longrightarrow Y$ such that $h_n : f \mid X^n \simeq \iota_n \circ g_n$, where $\iota_n : Y^n \longrightarrow Y$ is the inclusion. For an attaching map $j : S^n \longrightarrow X^n$ of a cell $\tilde{j} : \mathbb{D}^{n+1} \longrightarrow X$, we apply **HELP** to the following diagram, since $\iota_{n+1} : Y^n \longrightarrow Y^{n+1}$ is an $(n+1)$-equivalence according to last proposition



where $\iota : Y^n \longrightarrow Y^{n+1}$ is the inclusion. We could apply the same argument for an arbitrary family of attaching map. Since the $(n+1)$-skeleton is the pushout of attaching maps and inclusions $S^n \longrightarrow \mathbb{D}^{n+1}$, the maps and homotopies so far constructed induce the required map and homotopy on $(n+1)$-skeleton X^{n+1}. Note that, since the map g_n and the homotopy h_n are relative to A the map and homotopy g_{n+1}, h_{n+1} are also relative to A. The maps g_n induce the required cellular map on X. $\square$

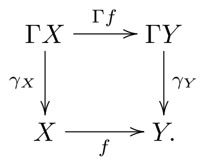
As consequence we have the following proposition.

Proposition 5.26. *Any map $f : X \longrightarrow Y$ between CW-complexes is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

Proof. The first statement is the direct consequence of last proposition. To see the second statement, observe that a homotopy $h : f \simeq g$ is a map $h : (X \times I, X \times \partial I) \longrightarrow (Y, Y)$, and vice versa. Applying last proposition to this map, gives the desired cellular homotopy. $\square$

We have a sequence of approximation theorems for spaces, pair of spaces and triads.

Theorem 5.27 (Approximation by CW-complexes). *For any space X, there is CW-complex ΓX and a weak equivalence $\gamma_X : \Gamma X \longrightarrow X$. For a map $f : X \longrightarrow Y$ and another such CW approximation $\gamma_Y : \Gamma Y \longrightarrow Y$, there is a map $\Gamma f : \Gamma X \longrightarrow \Gamma Y$, unique up to homotopy, such that the following diagram is homotopy commutative*



If X is n-connected, with $n \geq 1$, then ΓX can be chosen to have a unique vertex and no q-cells for $1 \leq q \leq n$.

Proof. First we show the existence of such approximation.It is sufficient to consider the case X path-connected, since the procedure can be applied to each path component separately.Again the construction is recursively.Fix a basepoint $y_0 \in Y$ and let X_1 be the wedge $\bigvee_{(q,j)} S^q, q \geq 1$, where $j : S^q \longrightarrow X$ represents a generator of the group $\pi_q(X)$.On the (q,j)th wedge summand, the map γ_1 is the given map j.X_1 has the obvious CW structure.The way we constructed X_1 shows that the induced homomorphism $\pi_q(\gamma_1)$ is surjective for all q.Assume that we have constructed CW-complexes X_{m-1}, cellular inclusions $i_m : X_{m-1} \longrightarrow X_m$ and maps $\gamma_m : X_m \longrightarrow X$ for $m \leq n$ such that $\gamma_m \circ i_{m-1} = \gamma_{m-1}$ and $(\gamma_m)_* : \pi_q(X_m) \longrightarrow \pi_q(X)$ is a surjection for all q and a bijection for $q < m$.We may attache $(n+1)$-cells to X_n to vanish the kernel of the map $(\gamma_n)_*$, and then extend over these cells to define a map

$$\gamma_{n+1} : X_{n+1} = X_n \cup (\cup_\alpha \mathbb{D}^{n+1}_\alpha) \longrightarrow X.$$

The cellular approximation theorem then implies that $\pi_q(\gamma_{n+1})$ is injective for $q \leq n$, and it is obviously surjective for all q.Now the CW approximation ΓX is the colimit of the inclusions $i_n : X_n \longrightarrow X_{n+1}$ and the weak equivalence $\gamma_X : \Gamma X \longrightarrow X$ is the induced map on ΓX from the maps $\gamma_n : X_n \longrightarrow X$.If X is n-connected, then we have used no q-cells for $q \leq n$ in the construction.The uniqueness and existence of Γf is immediate since the Whitehead theorem gives a bijection $(\gamma_Y)_* : [\Gamma X, \Gamma Y] \longrightarrow [\Gamma X, Y]$. □

Remark 5.28. This theorem say that the CW approximation is a functor Γ from the homotopy category of topological spaces to itself, and that there is natural transformation $\gamma : \Gamma \longrightarrow \mathrm{Id}$ such that the morphism $\gamma_X : \Gamma X \longrightarrow X$ is a weak equivalence.

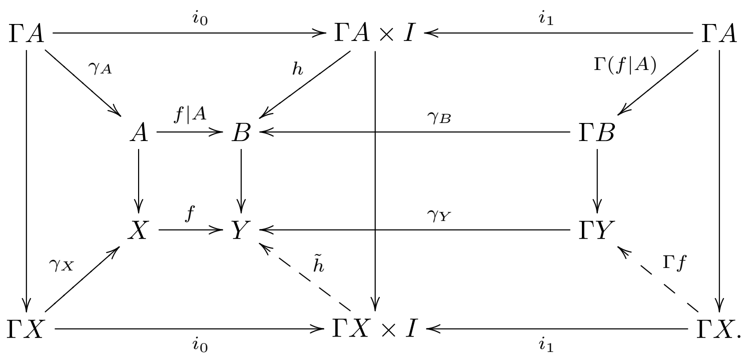
There is a relative generalization of last the theorem, as follows.

Theorem 5.29. *For any pair of spaces (X, A) and any CW approximation $\gamma_A : \Gamma A \longrightarrow A$, there is a CW approximation $\gamma_X : \Gamma \longrightarrow X$ such that ΓA is a subcomplex of ΓX and γ_X restricts to γ_A.If $f : (X, A) \longrightarrow (Y, B)$ is a map of pairs and $\gamma : (\Gamma Y, \Gamma B) \longrightarrow (Y, B)$ is another such CW approximation of pairs, there is a map $\Gamma f : (\Gamma X, \Gamma A) \longrightarrow (\Gamma Y, \Gamma B)$, unique up to homotopy, such that the following diagram of pairs is homotopy commutative*

$$\begin{array}{ccc} (\Gamma X, \Gamma A) & \xrightarrow{\Gamma f} & (\Gamma Y, \Gamma B) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ (X, A) & \xrightarrow[f]{} & (Y, B). \end{array}$$

If (X, A) is n-connected, then $(\Gamma X, \Gamma A)$ can be chosen to have no q-cells for $q \leq n$.

Proof. The idea of the construction of the pair $(\Gamma X, \Gamma A)$ is the same as that of last theorem, we have only to construct in the same fashion the CW approximation ΓX over the given approximation ΓA in order to guarantee that ΓA is a subcomplex of ΓX. To construct Γf, we first construct it on ΓA by mean of the last theorem, that is we have a map $\Gamma(f \mid A) : \Gamma A \longrightarrow \Gamma B$ by the last theorem, and then extend this map to all ΓX by use of **HELP**



The uniqueness up to homotopy of Γf is proved similarly.

□

Definition 5.30. *A* triad$(X; A, B)$ *is a space X together with subspaces A and B. A triad is said to be* excisive *if X is the union of the interiors of A and B. A* CW-triad $(X; A, B)$ *is a CW-complex X with subcomplexes A and B such that $X = A \cup B$.*

Remark 5.31. A traid $(X; A, B)$ must not be confused with a triple (X, A, B), which would require $B \subset A \subset X$.

Theorem 5.32. *If $e : (X; A, B) \longrightarrow (X'; A', B')$ is a map of excisive triads, such that the maps*

$$e : C \longrightarrow C', \quad e : A \longrightarrow A' \quad \textit{and} \quad e : B \longrightarrow B'$$

are weak equivalences, where $C = A \cap B$ and $C' = A' \cap B'$, then $e : X \longrightarrow X'$ is a weak equivalence.

Remark 5.33. A CW-triad $(X; A, B)$ is not excisive, since A and B are closed in X, but a simple argument shows that it is equivalent to an excisive triad.

More generally, suppose that maps $i : C \longrightarrow A$ and $j : C \longrightarrow B$ are given. Define the double mapping cylinder

$$M(i,j) = A \cup_i (C \times I) \cup_j B$$

to be the space obtained from $C \times I$ by attaching A to $C \times \{0\}$ along i and attaching B to $C \times \{1\}$ along j. Let $A \cup_C B$ denote the pushout of i and j and observe that we have a natural quotient map $q : M(i,j) \longrightarrow A \cup_C B$ by collapsing the cylinder, sending (c,t) to the image of c in the pushout.

The following proposition says more about the quotient map $q : M(i,j) \longrightarrow A \cup_C B$.

Proposition 5.34. *For a cofibration $i : C \longrightarrow A$ and any map $j : C \longrightarrow B$, the quotient map $q : M(i,j) \longrightarrow A \cup_C B$ is a homotopy equivalence.*

When i is a cofibration and j is an inclusion, with $X = A \cup B$ and $C = A \cap B$, we can think of q as giving a map of triads

$$q : (M(i,j); A \cup_i (C \times [0,2/3)), (C \times (1/3,1]) \cup_j B) \longrightarrow (A \cup_C B; A, B).$$

The domain is excisive, and q restricts to homotopy equivalences from the domain spaces and their intersection to the target subspaces A, B, and C.This applies when $(X; A, B)$ is CW-triad with $C = A \cap B$. Now we can state and prove the theorem of approximation of excisive triads by CW-triads

Theorem 5.35. *Let $(X; A, B)$ be an excisive triad and let $C = A \cap B$. Then there is a CW-triad $(\Gamma X; \Gamma A, \Gamma B)$ and a map of triads*

$$\gamma : (\Gamma X; \Gamma A, \Gamma B) \longrightarrow (X; A, B)$$

such that, with $\Gamma C = \Gamma A \cap \Gamma B$, the maps

$$\gamma : \Gamma C \longrightarrow C, \quad \gamma : \Gamma A \longrightarrow A, \quad \gamma : \Gamma B \longrightarrow B, \quad \text{and} \quad \gamma : \Gamma X \longrightarrow X$$

are all weak equivalences. If (A, C) is n-connected, then $(\Gamma A, \Gamma C)$ can be chosen to have no q-cells for $q \leq n$, and similarly for (B, C). Up to homotopy, CW approximation of excisive triads is functorial in such a way that γ is natural.

Proof. Chose a CW approximation $\gamma : \Gamma C \longrightarrow C$ and use the CW approximation of patrs to extend it to a CW approximation

$$\gamma : (\Gamma A, \Gamma C) \longrightarrow (A, C) \qquad \text{and} \qquad \gamma : (\Gamma B, \Gamma C) \longrightarrow (B, C).$$

We then define ΓX to be the pushout $\Gamma A \cup_{\Gamma C} \Gamma B$ and let $\gamma : \Gamma X \longrightarrow X$ be given by the universal property of pushouts. The way we have constructed ΓA

and ΓB shows at once that $\Gamma C = \Gamma A \cap \Gamma B$. To see that $\gamma : \Gamma X \longrightarrow X$ is a weak equivalence, define $X'' = M(i,j)$, where $i : \Gamma C \longrightarrow \Gamma A$ and $j : \Gamma C \longrightarrow \Gamma B$ are the inclusions of subcomplexes, and let $A'' = \Gamma A \cup_i (\Gamma C \times [0, 2/3))$ and $B'' = (\Gamma C \times (1/3, 1]) \cup_j \Gamma B$ and $C'' = A'' \cap B''$, then we know that the map $q : (X''; A'', B'') \longrightarrow (\Gamma X; \Gamma A, \Gamma B)$ described above is a homotopy equivalence. Now define $e : (X''; A'', B'') \longrightarrow (X; A, B)$ to be the composite $e = \gamma \circ q$, this satisfies the conditions of the Theorem 5.32, thus $e : X'' \longrightarrow X$ is a weak equivalence which implies that $\gamma : \Gamma X \longrightarrow X$ is too. Other affirmations are direct consequences of the theorem for pairs. This ends the proof. $\square$

6 The Homotopy Excision And Suspension Theorems

Definition 6.1. *A map* $f : (A, C) \longrightarrow (X, B)$ *of pairs is an* n*-equivalence,* $n \geq 1$*, if*

$$(f_*^{-1})(im(\pi_o(B) \longrightarrow \pi_0(X))) = im(\pi_0(C) \longrightarrow \pi_0(A))$$

and for all choices of basepoints in C*,*

$$f_* : \pi_q(A, C) \longrightarrow \pi_q(X, B)$$

is a bijection for $q < n$ *and a surjection for* $q = n$*.*

Remark 6.2. The first condition in the definition holds whenever A and X are path connected.

Proposition 6.3. *For a triple* (X, A, B) *and any basepoint in* B*, the following sequence is exact*

$$\cdots \longrightarrow \pi_q(A, B) \xrightarrow{i_*} \pi_q(X, B) \xrightarrow{j_*} \pi_q(X, A) \xrightarrow{k_* \circ \partial} \pi_{q-1}(A, B) \longrightarrow \cdots$$

where

$$i : (A, B) \longrightarrow (X, B), \;\; j : (X, B) \longrightarrow (X, A) \;\; \text{and} \;\; k : (A, *) \longrightarrow (A, B)$$

are inclusions.

Proof. The proof is a purely algebraic argument on the exact sequences of pairs, for instance to show the exactness of

$$\pi_q(A, B) \xrightarrow{i_*} \pi_q(X, B) \xrightarrow{j_*} \pi_q(X, A)$$

one has only to chase the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_q(B) & \xrightarrow{j_*} & \pi_q(A) & \longrightarrow & \pi_q(A, B) & \longrightarrow & \pi_{q-1}(B) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle id} & \\
\cdots \longrightarrow & \pi_q(B) & \longrightarrow & \pi_q(X) & \longrightarrow & \pi_q(X, B) & \longrightarrow & \pi_{q-1}(B) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & \\
\cdots \longrightarrow & \pi_q(A) & \xrightarrow[i_*]{} & \pi_q(X) & \longrightarrow & \pi_q(X, A) & \longrightarrow & \pi_{q-1}(A) & \longrightarrow \cdots .
\end{array}$$

Note that the sequence in question is represented vertically in this diagram. □

Definition 6.4. *For a triad $(X; A, B)$ with basepoint $* \in C = A \cap B$, define*

$$\pi_q(X; A, B) = \pi_{q-1}(P(X; *, B), P(A; *, C))$$

where $q \geq 2$.

Remark 6.5. More explicitly, $\pi_q(X; A, B)$ is the set of homotopy classes of maps of tetrads

$$(I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-1} \times I^{q-1} \times \{0\}) \longrightarrow (X; A, B, *)$$

where again $J^{q-1} = \partial I^{q-2} \times I \cup I^{q-2} \times \{0\}$.

Remark 6.6. The long exact sequence of the pair $(P(X; *, B), P(A; *, C))$ is the following sequence, which is clear from the definitions of homotopy groups of pairs and triads

$$\cdots \longrightarrow \pi_q(A, C) \longrightarrow \pi_q(X, B) \longrightarrow \pi_q(X; A, B) \longrightarrow \cdots .$$

The following theorem has many consequences and applications especially in homology theory. One of its consequences is the Freudenthal suspension theorem, which is the starting point of the stable homotopy theory.

Theorem 6.7 (Homotopy Excision). *Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is not empty. Assume that (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected, where $m \geq 2$ and $n \geq 1$. Then the inclusion $(A, C) \longrightarrow (X, B)$ is an $(m+n-2)$-equivalence.*

Sketch of Proof. We admit the following technical result and show that it is equivalent to the homotopy excision theorem. Under the hypothesis of the excision theorem we have

$$\pi_q(X; A, B) = 0 \quad \text{for} \quad 2 \leq q \leq m+n-2$$

and all choices of basepoints in C. Now we return to the homotopy excision theorem. The conditions that $m \geq 1$ and $n \geq 1$ give the sujectivity of the maps $\pi_0(C) \longrightarrow \pi_0(A)$ and $\pi_0(C) \longrightarrow \pi_0(B)$, so it remains to show that the induced map $f_* : \pi_q(A, C) \longrightarrow \pi_q(X, B)$ is a bijection for $q < n$ and a surjection for $q = n$. The fact that $m \geq 2$ implies that (A, C) is 1-connected, that is

$$\pi_1(A, C) = \pi_0(P(A; *, C)) = 0.$$

It is now easy to see that this implies also that (X, B) is 1-connected. The idea is to replace any fragment of paths that lie in A by a path that is entirely in C and this is possible since (A, C) is 1-connected. This shows that (X, B)

is 1-connected, since $(X; A, B)$ is excisive. The Remark 6.6 says exactly that $\pi_q(A, C) \cong \pi_q(X, B)$ if and only if $\pi_{q+1}(X; A, B) = 0$ and

$$\pi_n(f) : \pi_n(A, C) \longrightarrow \pi_n(X, B)$$

is surjective if and only if $\pi_n(X; A, B) = 0$, hence the last result is equivalent to the homotopy excision theorem. □

Now we see some applications of this theorem.

Theorem 6.8. *Let $f : X \longrightarrow Y$ be an $(n-1)$-equivalence between $(n-2)$-connected spaces, where $n \geq 2$. Then the quotient map $\pi : (M_f, X) \longrightarrow (C_f, *)$ is a $(2n-2)$-equivalence, where C_f is the unreduced cofiber M_f/X. In particular, C_f is $(n-1)$-connected. If X and Y are $(n-1)$-connected, then $\pi : (M_f, X) \longrightarrow (C_f, *)$ is $(2n-1)$-equivalence.*

Proof. We have the excisive triad$(C_F; A, B)$, where

$$A = Y \cup (X \times [0, 2/3]) \qquad \text{and} \qquad B = (X \times [1/3, 1])/(X \times \{1\}).$$

Thus $C := A \cap B = X \times [1/3, 2/3]$. It is easy to check that π is homotopic to the composite

$$(M_f, X) \longrightarrow (A, C) \xrightarrow{\theta} (C_f, B) \longrightarrow (C_f, *),$$

where the first and last maps are homotopy equivalences of pairs. From Proposition 1.10 f can be written as the composite $X \xrightarrow{j} M_f \xrightarrow{r}$, where r is a homotopy equivalence, thus $\pi_q(r)$ is an isomorphism and $\pi_q(r \circ j) = \pi_q(r) \circ \pi_q(j) = \pi_q(f)$ for all q. So $\pi_q(f)$ is an epimorphism(or isomorphism) if and only if $\pi_q(j)$ is an epimorphism(or isomorphism). Now consider the exact sequence of the pair (M_f, X)

$$\cdots \longrightarrow \pi_q(X) \xrightarrow{\pi_q(j)} \pi_q(M_f) \longrightarrow \pi_q(M_f, X) \longrightarrow \pi_{q-1}(X) \longrightarrow \cdots$$

if $q \leq n-1$ then $\pi_q(f)$ and therefore $\pi_q(j)$ are epimorphisms, and since X is $(n-2)$-connected this exact sequence shows that $\pi_q(Mf, X)$ is the trivial group. Thus (M_f, X) is $(n-1)$-connected, this together with the homotopy equivalence $(M_f, X) \cong (A, C)$ show at once that (A, C) is $(n-1)$-connected as well. The exact sequence of the pair (CX, X) and the fact that the cone CX is contractible(and therefore $\pi_*(CX) = 0$), show that the connecting homomorphism $\partial : \pi_{q+1}(CX, X) \longrightarrow \pi_q(X)$ is in fact an isomorphism, thus (CX, X) is $(n-1)$-connected, and it is n-connected if X is $(n-1)$-connected. There is an evident homotopy equivalence $(CX, X) \cong (B, C)$, hence (B, C) is $(n-1)$-connected, and it is n-connected if X is $(n-1)$-connected. Now the result follows from the homotopy excision theorem. □

Theorem 6.9. *Let $i : A \longrightarrow X$ be a cofibration and an $(n-1)$-equivalence between $(n-2)$-connected spaces, where $n \geq 2$. Then the quotient map $(X, A) \longrightarrow (X/A, *)$ is a $(2n-2)$-equivalence, and it is a $(2n-1)$-equivalence if A and X are $(n-1)$-connected.*

Proof. The verticals arrows are homotopy equivalences of pairs in the following commutative diagram

$$\begin{array}{ccc} (M_i, A) & \xrightarrow{\ \pi\ } & (C_i, *) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \psi} \\ (X, A) & \longrightarrow & (X/A, *) \end{array}$$

where $r : (M_i, A) \longrightarrow (X, A)$ is the retraction defined in section 1, and $\psi : (C_i, *) \longrightarrow (X/A, *)$ is the homotopy equivalence given in Proposition 3.22 .The result follows from the previous theorem. $\square$

Definition 6.10. *For a based space X, define the* suspension homomorphism

$$\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$$

by letting

$$\Sigma f = f \wedge id : S^{q+1} \cong S^q \wedge S^1 \longrightarrow X \wedge S^1 = \Sigma X.$$

Remark 6.11.

A map $f \in \pi_q(X)$ can be regarded as a map of pairs $f : (I^q, \partial I^q) \longrightarrow (X, *)$, then the product $f \times id : I^{q+1} = I^q \times I \longrightarrow X \times I$ passes to quotients to give a map of triples

$$(I^{q+1}, \partial I^{q+1}, J^{q+1}) \longrightarrow (C'X, X, *)$$

*whose restriction to $I^q \times \{1\}$ is f and which induces Σf when we quotient out $X \times \{1\}$. Note that we see X as the subspace $X \times \{1\}$ of $C'X$.*We can define more explicitly Σ as follows. Let $C'X$ be the reversed cone on X, that is the space

$$C'X = X \times I/(X \times \{0\} \cup \{*\} \times I).$$

A map $f \in \pi_q(X)$ can be regarded as a map of pairs $f : (I^q, \partial I^q) \longrightarrow (X, *)$, then the product $f \times \mathrm{id} : I^{q+1} = I^q \times I \longrightarrow X \times I$ passes to quotients to give a map of triples

$$(I^{q+1}, \partial I^{q+1}, J^{q+1}) \longrightarrow (C'X, X, *)$$

whose restriction to $I^q \times \{1\}$ is f and which induces Σf when we quotient out $X \times \{1\}$. Note that we see X as the subspace $X \times \{1\}$ of $C'X$.

Theorem 6.12 (Freudenthal Suspension). *Assume that X is nondegenerately base and$(n-1)$-connected, where $n \geq 1$. Then the suspension homomorphism Σ is a bijection if $q < 2n-1$ and a surjection if $q = 2n-1$.*

Proof. According to the last remark the following diagram is commutative, where $\rho : C'X \longrightarrow \Sigma X$ is the quotient map

$$\begin{array}{ccc} & \pi_{q+1}(C'X, X, *) & \\ {}^{\partial}\swarrow & & \searrow^{\rho_*} \\ \pi_q(X) & \xrightarrow[\Sigma]{\hspace{3em}} & \pi_{q+1}(\Sigma X). \end{array}$$

As we have seen in the proof of Theorem 6.8, since $C'X$ is contractible, ∂ is an isomorphism. Up to degree $n-1$, both of $\pi_q(X)$ and $\pi_q(C'X)$ are trivial and $\pi_n(C'X) = 0$, so the inclusion $X \longrightarrow C'X$ is an n-equivalence between $(n-1)$-connected spaces. From Proposition 3.17 we know that this inclusion is a cofibration, therefore the quotient map

$$\rho : (C'X, X) \longrightarrow (C'X/X, *) \cong (\Sigma X, *)$$

is a $2n$-equivalence by the last theorem. The conclusion now follows from the bijectivity of ∂ and the last diagram. □

Proposition 6.13. *$\pi_1(S^1) = \mathbb{Z}$ and $\Sigma : \pi_1(S^1) \longrightarrow \pi_2(S^2)$ is an isomorphism.*

Theorem 6.14. *For all $n \geq 2$, $\pi_n(S^n) = \mathbb{Z}$ and $\Sigma : \pi_n(S^n) \longrightarrow \pi_{n+1}(S^{n+1})$ is an isomorphism.*

Proof. S^n is an $(n-1)$-connected space by Proposition 4.20 and since $n > 1$ the Freudenthal theorem applies and implies the second statement. For the first statement, we use the last proposition. □

Definition 6.15. *The* q-th stable homotopy group $\pi_q^s(X)$ *of a space X, is the colimit of the following diagram*

$$\pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X) \longrightarrow \pi_{q+2}(\Sigma^2 X) \longrightarrow \cdots .$$

Remark 6.16. We can show by induction on $n \geq 1$ and using the argument and diagram in the proof of Freudenthal theorem, that the space $\Sigma^n(X)$ is $(n-1)$-connected,thus for $q < n-1$ there is an isomorphism

$$\pi_{q+n}(\Sigma^n X) \cong \pi_{q+n+1}(\Sigma^{n+1} X)$$

by Freudenthal theorem, and therefore for any fixed q, we have

$$\pi_{q+n}(\Sigma^n X) \cong \pi_{q+n+1}(\Sigma^{n+1} X) \cong \pi_{q+n+2}(\Sigma^{n+2} X) \cong \cdots$$

for all $n > q+1$. This implies that the q-th stable homotopy group is isomorphic to the group $\pi_{q+n}(\Sigma^n X)$ for $n > q+1$.

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