Field and Galois Theory

Problems Set 4

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In the following problems, K/F is a Galois extension, with Galois group Γ.

Definition. A *profinite group* is an inverse limit of a projective (inverse) system of finite groups. Let $G = \varprojlim G_\alpha$ be a profinite group, with $\{G_\alpha\}$ a projective system of finite groups. Then G, being a subgroup of the product $\prod_\alpha G_\alpha$, inherits the subspace topology, where we endow each finite group G_α with the discrete topology and the product with the product topology. This is the natural topology of a profinite group, which makes it a topological group.

1) Let G be a topological group. Show that if G is profinite then G is Hausdorff, compact and has a system of open neighborhoods of 1 consisting of normal subgroups. Conversely, show that if G is Hausdorff, compact and has a system of open neighborhoods of 1 consisting of normal subgroups, then the projections induce an isomorphism of topological groups $G \cong \varprojlim G/N_\alpha$ where $\{N_\alpha\}_\Lambda$ is any directed family[1] of open normal subgroups of G with $\cap_{\alpha\in\Lambda} N_\alpha = \{1\}$[2], and therefore, G is profinite.

2) Show that Γ is profinite by proving the isomorphism of topological groups:
$$\Gamma \cong \varprojlim_{\alpha\in\Lambda} \mathrm{Gal}(L_\alpha/F)$$

[1] This means that for every α and β, there exists a γ such that $N_\gamma \subseteq N_\alpha \cap N_\beta$.
[2] why does such a family exist?

where $\{L_\alpha\}_\Lambda$ is any directed family[3] of finite Galois subextensions L_α/F such that $K = \cup_{\alpha\in\Lambda} L_\alpha$

3) Let G be a compact topological group and $H \leq G$ a subgroup. Show that:

   (a) H is open in G if and only if H is closed in G and $[G : H] < \infty$.

   (b) If H is open, then the intersection of all conjugates of H in G is an open normal subgroup of G.

   (c) If G is profinite and H is closed and normal, then G/H is profinite as well.

4) Let $H \leq \Gamma$, prove that $\bar{H} = \bigcap_{N\in\mathcal{N}} HN$.

5) Let L be a be subextension of K/F, show that the subspace topology of $\mathrm{Gal}(K/L)$ induced by the Krull topology of Γ is equivalent to the Krull topology of $\mathrm{Gal}(K/L)$.

6) Let X be a finite discrete continuous Γ-set, i.e. the action $\Gamma \times X \to X$ is continuous where X has the discrete topology and Γ has the Krull topology. Show that there exists a finite Galois subextension L/F such that the action factors through $\mathrm{Gal}(L/F)$, i.e. $\Gamma \to \mathrm{Sym}(X)$ factors through $\mathrm{Gal}(L/F)$.

7) Let $\mathbb{Q}^{ab}$ be the extension of $\mathbb{Q}$ obtained by adjoining all roots of unity and $\mathbb{Q}(\zeta_{p^\infty})$ the extension of $\mathbb{Q}$ obtained by adjoining all p^n-th roots of 1 (i.e., $\mathbb{Q}(\zeta_{p^\infty}) = \mathbb{Q}(\{\zeta \in \bar{\mathbb{Q}} \mid \exists n \geq 1, \zeta^{p^n} = 1\})$). Show that:

   (a) The extension $\mathbb{Q}^{ab}/\mathbb{Q}$ is Abelian and there exists an isomorphism of topological groups

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) = (\hat{\mathbb{Z}})^\times$$

   (b) The extension $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$ is Abelian and there exists an isomorphism of topological groups

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times$$

[3] This means that for every α and β, there exists a γ such that $L_\alpha L_\beta \subseteq L_\gamma$

(c) There exists a canonical decomposition of topological rings:

$$\hat{\mathbb{Z}} \cong \Pi_p \mathbb{Z}_p$$

(d) The following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d, "r_p"'] & (\hat{\mathbb{Z}})^\times \arrow[d, "\pi_p"] \\
\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \arrow[r, "\cong"'] & \mathbb{Z}_p^\times
\end{tikzcd}$$

where the isomorphisms are given by (a) and (b), the projection π_p is given by the decomposition in (c) and r_p is given by the restriction.